17. Lecture 17 (Apr 7): Differentials

Recommended reading: Hartshorne II 8, Kempf §6.3

17.1. Review of Kähler differentials

To a map of rings $A \to B$ we associate the B-module $\Omega^1_{B/A}$ of **Kähler differentials** of B over A. It is the target of an A-linear derivation

$$d: B \longrightarrow \Omega^1_{B/A}$$

(i.e. an A-module map into a B-module satisfying the Leibniz rule $d(fg) = f \cdot d(g) + g \cdot d(f)$) which is universal in the sense that every A-linear derivation $\delta: B \to M$ into a B-module M factors as $\delta = \bar{\delta} \circ d$ for a unique B-module map $\bar{\delta}: \Omega^1_{B/A} \to M$. For completely formal reasons, $\Omega^1_{B/A}$ is generated as a B-module by the elements df $(f \in B)$ subject to the relations

$$d(f+g) = df + dg, \qquad d(ff') = fdg + gdf, \qquad da = 0 \qquad (f, f' \in B, a \in A).$$

Applying these inductively, we deduce that for every polynomial $f \in A[T]$ and $b \in B$, we have $d(f(b)) = f'(b)db$ where f' is the formal derivative of f.

If $B = A[T_1, \ldots, T_n]/(f_1, \ldots, f_r)$ is a (finite) presentation of B as an A-algebra, then $\Omega^1_{B/A}$ is generated by the finite number of elements dT_i subject to the relations

$$0 = df_j = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i} dT_i,$$

or equivalently can be written as the cokernel of the "Jacobian matrix" $J = [\partial f_j / \partial T_i]$ treated as a map $B^r \to B^n$.

The following proposition lists the most important properties of $\Omega^1_{B/A}$. It is phrased in the less-than-optimal way, postulating the existence of certain canonical maps instead of constructing them. See the subsequent remark for extra precision.

Proposition 17.1.1. *Let $A \to B$ be a map of rings.*

(a) (Base change) Let $A \to A'$ be a map of rings and let $B' = B \otimes_A A'$. Then

$$\Omega^1_{B'/A'} \simeq \Omega^1_{B/A} \otimes_B B'.$$

(b) (Transitivity) For every map of rings $B \to C$ we have an exact sequence of C-modules

$$\Omega^1_{B/A} \otimes_A C \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0.$$

(c) (Localization) For every multiplicative system $S \subseteq B$, we have

$$\Omega^1_{B[S^{-1}]/A} \simeq \Omega^1_{B/A}[S^{-1}].$$

(d) (Conormal sequence) For every ideal $I \subseteq B$ we have an exact sequence of B/I-modules

$$I/I^2 \xrightarrow{d} \Omega^1_{B/A} \otimes_B B/I \longrightarrow \Omega^1_{(B/I)/A} \longrightarrow 0.$$

(e) (Characterization in terms of the diagonal) Let J be the kernel of the surjective map $B \otimes_A B \to B$, $b \otimes b' \mapsto bb'$. Then

$$\Omega^1_{B/A} \simeq J/J^2.$$

Remark 17.1.2. Construction of the various maps appearing above:

(a) Since $B \to B' \to \Omega^1_{B'/A'}$ is a B-linear derivation, by the universal property of $\Omega^1_{B/A}$ we obtain a B-linear map $\Omega^1_{B/A} \to \Omega^1_{B'/A'}$, which by adjunction between forgetful and base change corresponds to a B''-linear map $\Omega^1_{B/A} \otimes_B B' \to \Omega^1_{B'/A'}$.

(b) The first map in the sequence is adjoint to the A-module map $\Omega^1_{B/A} \to \Omega^1_{C/A}$ corresponding to the A-linear derivation $B \to C \to \Omega^1_{C/A}$. The second one corresponds to the A-linear derivation $C \to \Omega^1_{C/B}$.

(c) To get the map $\Omega^1_{B/A}[S^{-1}] \to \Omega^1_{B[S^{-1}]/A}$, apply the construction in (b) to $C = B[S^{-1}]$.

(d) The second map is again (b), this time applied to $C = B/I$. The first one is the unique map making the diagram below commute

$$\begin{tikzcd}
I \arrow[r, "d|_I"] \arrow[d] & \Omega^1_{B/A} \arrow[d] \\
I/I^2 \arrow[r, dotted] & \Omega^1_{B/A} \otimes_B B/I
\end{tikzcd}$$

(since the left map is surjective, the desired map is unique if it exists. One still needs to check that it does.)

(e) The morphism $\Omega^1_{B/A} \to J/J^2$ is the unique B-module map sending db to the class of $b \otimes 1 - 1 \otimes b$.

Proof of Proposition 17.1.1. See Stacks Project: (a) [SP Lemma 00RV] (b) [SP Lemma 00RS] (c) [SP Lemma 00RT] (d) [SP Lemma 00RU] (e) [SP Lemma 00RW]. ☐

17.2. Sheaves of differentials

Let X be an algebraic set. We can construct a coherent $\mathcal{O}_X$-module Ω^1_X together with a k-linear derivation

$$d : \mathcal{O}_X \longrightarrow \Omega^1_X$$

in either of the following ways:

(a) as the universal k-linear derivation into a *quasi-coherent* $\mathcal{O}_X$-module;

(b) as the sheaf whose value on an affine open $U \subseteq X$ is $\Omega^1_{\mathcal{O}(U)/k}$ (this works thanks to Proposition 17.1.1(c));

(c) as $\mathcal{I}_\Delta/\mathcal{I}_\Delta^2$ where $\mathcal{I}_\Delta \subseteq \mathcal{O}_{X\times X}$ is the ideal sheaf of the diagonal $X \simeq \Delta \subseteq X \times X$ (assuming that X is separated).

We omit the details — see the recommended references or Stacks Project, section 01UM.

Remark 17.2.1. In basically the same way, we define $\Omega^1_{X/S}$ for a map of schemes $X \to S$. The resulting sheaf is always quasi-coherent, and is coherent if S is locally noetherian and $X \to S$ is locally of finite type.

Proposition 17.2.2. *A variety X is smooth if and only if Ω^1_X is locally free.*

Proof. The key point of the argument is that the fiber $\Omega^1_X(x)$ at a point $x \in X$ is isomorphic to the Zariski cotangent space ${}_x/{}^2_x$ of the local ring $\mathcal{O}_{X,x}$. Assuming this, X is smooth iff the dimension of ${}_x/{}^2_x = \Omega^1_X(x)$ is constant, which happens precisely when Ω^1_X is locally free. To see the claim (which we've already seen in some other form), we note that for a k-vector space V, considered as an $\mathcal{O}_{X,x}$-module via $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x = k$, k-linear maps $\mathfrak{m}_x/\mathfrak{m}_x^2 \to V$ correspond to k-linear derivations $\mathcal{O}_{X,x} \to V$. Thus the k-vector spaces $\Omega^1_X(x)$ and $\mathfrak{m}_x/\mathfrak{m}_x^2$ represent the same functor and hence are isomorphic. □

Our first (non-affine) example is the projective space.

Theorem 17.2.3 (Euler sequence)**.** *On $X = \mathbb{P}^n$, there is a short exact sequence of coherent sheaves*

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{\mathbb{P}^n} \arrow[r] & \mathcal{O}(-1)^{n+1} \arrow[r, "(x_0,\dots,x_n)"] & \mathcal{O} \arrow[r] & 0.
\end{tikzcd}$$

Proof. The key point behind the construction of the left map in the sequence is that for a homogeneous f of degree d, the partial derivatives $\partial f/\partial x_i$ are homogeneous of degree $d-1$. For $i = 0,\dots,n$ we define a k-linear derivation $\partial_i \colon \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)$ as follows. For an open $U \subseteq \mathbb{P}^n$ of the form $U = D(f)$ for a homogeneous $f \in k[x_0,\dots,x_n]$ of degree d, we have

$$\mathcal{O}(U) = k[x_0,\dots,x_n][f^{-1}]_0$$

and we define $\partial_i(g/f^m)$ (where $g \in k[x_0,\dots,x_n]_{md}$) by the standard formula

$$\partial_i\left(\frac{g}{f^m}\right) = \frac{\partial(g/f^m)}{\partial x_i} = \frac{\left(\frac{\partial g}{\partial x_i}\right) f - mg\left(\frac{\partial f}{\partial x_i}\right)}{f^m},$$

which is a homogeneous element of $k[x_0,\dots,x_n][f^{-1}]$ of degree -1, i.e. an element of $\Gamma(U, \mathcal{O}_{\mathbb{P}^n}(-1))$. It is trivial to check that these maps glue compatibly and that they give rise to a k-linear derivation.

Taken together, the maps ∂_i give a k-linear derivation $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}$, and hence a map of coherent sheaves

$$\Omega^1_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}.$$

In order to check the exactness of the resulting sequence, it suffices to check on the standard opens. On $U_0 = D(x_0)$, the sequence takes the form

$$\begin{tikzcd}
\bigoplus_{i=1}^n k[y_1,\dots,y_n]dy_i \arrow[r] & k[y_1,\dots,y_n]e_0 \oplus \bigoplus_{i=1}^n k[y_1,\dots,y_n]e_i \arrow[r] & k[y_1,\dots,y_n]
\end{tikzcd}$$

where the first map sends dy_i to $e_i - x_i e_0$ and the second sends $e_0 \mapsto 1$ and $e_i \mapsto x_i$ for $i > 0$. The image of the first and the image of the second are both equal to the set of $f_0 e_0 + \sum_{i=1}^n f_i e_i$ satisfying $f_0 = -\sum_{i=1}^n x_i f_i$. The rest is clear. □

Proposition 17.2.4 (Conormal sequence)**.** *Let X be a smooth variety and let $Y \subseteq X$ be a smooth divisor (codimension one subvariety). Then $\mathcal{I}_Y \simeq \mathcal{O}_X(-Y)$, and we have a short exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(-Y)|_Y \arrow[r] & \Omega^1_X|_Y \arrow[r] & \Omega^1_Y \arrow[r] & 0.
\end{tikzcd}$$

Proof. Without injectivity on the left, this is the global version of the conormal sequence for Kähler differentials (see Proposition 17.1.1(d)). In order to check that the left map is injective, it is enough to check it on fibers at all $y \in Y$. Let $f \in \mathcal{O}_{X,y}$ be a local generator of $\mathcal{I}_Y$. Then $f \in \mathfrak{m}_y \setminus \mathfrak{m}_y^2$, and hence its image in $\Omega^1_X|_Y(y) = \Omega^1_X(y) = \mathfrak{m}_y/\mathfrak{m}_y^2$ is nonzero. □

17.3. The canonical bundle

Definition 17.3.1. Let X be a smooth algebraic variety.

(a) For $p \geq 0$ we define Ω_X^p to be the p-th exterior power of Ω_X^1. (By convention, $\Omega_X^0 = \mathcal{O}_X$.) We call Ω_X^p the sheaf of **differential p-forms**.

(b) We call the dual $\mathcal{H}om(\Omega_X^1, \mathcal{O}_X) = (\Omega_X^1)^\vee$ the **tangent sheaf** of X and denote it by $\mathcal{T}_X$.

(c) Suppose that $\dim(X) = n$. Then $\omega_X = \Omega_X^n$ is an invertible sheaf, called the **canonical sheaf** of X.

In the special case when X is a curve, we have $\omega_X = \Omega_X^1$.

The computation of ω_X using various short exact sequences is enabled by the following linear algebra fact. For a locally free $\mathcal{O}_X$-module $\mathcal{E}$ on a ringed space X, let us define $\det(\mathcal{E}) = \bigwedge^r(\mathcal{E})$ where $r = \operatorname{rk}(\mathcal{E})$. Thus $\omega_X = \det(\Omega_X^1)$.

Lemma 17.3.2. *Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a sequence of locally free sheaves on a ringed space X. Then*

$$\det(\mathcal{F}) \simeq \det(\mathcal{E}) \otimes \det(\mathcal{G}).$$

Proof. If X is a point and $\mathcal{O}(X)$ is a field, this lemma postulates an isomorphism between two one-dimensional vector spaces. If we want to globalize this, we need a canonical such isomorphism which would also work for free modules over a ring. Intuitively, the assertion is that choosing a volume form on a vector space is equivalent to choosing one on a subspace and on the quotient space, a Fubini-type result.

If $u_1, \cdot, u_e$ is a basis of $\mathcal{E}$ and $w_1, \ldots, w_g$ a basis of $\mathcal{G}$, then $u = u_1 \wedge \cdots \wedge u_e$ and $w = w_1 \wedge \cdots \wedge w_g$ are basis elements of $\det(\mathcal{E})$ and $\det(\mathcal{G})$, respectively. Further, if $v_1, \ldots, v_g \in \mathcal{F}$ are elements lifting $w_1, \ldots, w_g$, then $u_1, \ldots, u_e, v_1, \ldots, v_g$ is a basis of $\mathcal{F}$ and $v = u_1 \wedge \cdots \wedge u_e \wedge v_1 \wedge \cdots \wedge v_g$ is a basis of $\det(\mathcal{F})$. The postulated isomorphism maps $u \otimes w$ to v. We check that different choices lead to the same isomorphism. If $u'_1, \ldots, u'_e$ and $w'_1, \ldots, w'_g$ are different choices of bases, then $u' = \det(A)u$ and $w' = \det(B)w$ where A and B are the respective change of base matrices. At the same time, we have $v' = \det(C)v$ where C is a change of base matrix from $u_1, \ldots, u_e, v_1, \ldots, v_g$ to $u'_1, \ldots, u'_e, v'_1, \ldots, v'_g$ for some lifts $v'_1, \ldots, v'_g$ of $w'_1, \ldots, w'_g$. The matrix C has a block-triangular form

$$C = \begin{bmatrix} A & * \\ 0 & B \end{bmatrix}$$

and hence $\det(C) = \det(A) \cdot \det(B)$. The exact same calculation works for a short exact sequence of free modules over a commutative ring.

For the globalization step, consider the diagram

$$\begin{tikzcd}
& \bigwedge^e \mathcal{E} \otimes \bigwedge^g \mathcal{F} \arrow[dl, "1\otimes q"'] \arrow[dr] & \\
\bigwedge^e \mathcal{E} \otimes \bigwedge^g \mathcal{G} \arrow[d, equal] & & \bigwedge^{e+g} \mathcal{F} \arrow[d, equal] \\
\det(\mathcal{E}) \otimes \det(\mathcal{G}) \arrow[rr, dotted] & & \det(\mathcal{F}).
\end{tikzcd}$$

The left arrow is the identity of $\bigwedge^e \mathcal{E}$ tensored with the surjection $\bigwedge^g \mathcal{F} \to \bigwedge^g \mathcal{G}$ and hence is surjective. The right arrow is given by exterior multiplication. We claim that there exists a unique dotted arrow making the diagram commute, and that it is an isomorphism. Uniqueness is clear (since the left map is surjecive), and then existence (and being an isomorphism) can be checked locally. We may therefore assume that $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ are free, in which case we can use the preceding discussion. □

Example 17.3.3 (The canonical sheaf of $\mathbb{P}^n$)**.** Let us compute $\omega_{\mathbb{P}^n}$ using the Euler sequence. Lemma 17.3.2 immediately yields

$$\omega_{\mathbb{P}^n} = \det(\Omega^1_{\mathbb{P}^n}) = \det(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}) \otimes \det(\mathcal{O}_{\mathbb{P}^n})^{-1} = \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

Example 17.3.4 (Adjunction formula)**.** Let Y be a smooth divisor on a smooth variety X. Then the conormal sequence gives

$$\omega_Y = \det(\Omega^1_Y) = \det(\Omega^1_X|_Y) \otimes \det(\mathcal{O}_X(-Y)|_Y)^{-1} = \omega_X|_Y \otimes \mathcal{O}_X(Y)|_Y = \omega_X(Y)|_Y.$$

For example, for a degree d plane curve $Y \subseteq \mathbb{P}^2$, the canonical sheaf is

$$\omega_Y \simeq \mathcal{O}_{\mathbb{P}^2}(d-3)|_Y.$$

For $d = 3$ this gives $\omega_Y \simeq \mathcal{O}_Y$.

17.4. Serre duality

For now, we simply state the Serre duality theorem for curves.

Theorem 17.4.1 (Serre duality)**.** *Let X be a smooth projective curve.*

(a) There is a canonical isomorphism $H^1(X, \omega_X) \simeq k$.

(b) For every invertible sheaf $\mathcal{L}$ on X, the bilinear map

$$H^0(X, \mathcal{L}^\vee \otimes \omega_X) \times H^1(X, \mathcal{L}) = \mathrm{Hom}(\mathcal{L}, \omega_X) \times H^1(X, \mathcal{L}) \longrightarrow H^1(X, \omega_X) \simeq k$$

(defined by functoriality of $H^1(X, -)$) is a perfect pairing, i.e. it gives an isomorphism

$$H^1(X, \mathcal{L}) \simeq H^0(X, \mathcal{L}^{-1} \otimes \omega_X)^\vee.$$

Definition 17.4.2. Let X be a smooth projective curve. We define its **genus** as the number

$$g(X) = \dim H^0(X, \omega_X) = \dim H^1(X, \mathcal{O}_X).$$

Example 17.4.3. For a plane curve $Y \subseteq \mathbb{P}^2$ of degree d, we have $g(Y) = (d-1)(d-2)/2$.

17.5. Problem session (Apr 7)

During the problem session:

1. We discussed **Serre's theorem** (every coherent sheaf on $\mathbb{P}^n$ is a quotient of $\mathcal{O}(-d)^m$ for some d and m) and how it implies that $H^q(X, \mathcal{F})$ is finite dimensional for a coherent sheaf $\mathcal{F}$ on a projective variety X. (This is covered in detail in the last section of Lecture 16.) We mentioned the fact (**Hilbert's syzygy theorem**) that every $\mathcal{F}$ on $\mathbb{P}^n$ admits a *finite* resolution by direct sums of $\mathcal{O}(d_i)$'s.

2. This let us to define the **Euler characteristic** of a coherent sheaf $\mathcal{F}$ on a projective variety X as

   $$\chi(\mathcal{F}) = \sum_{q=0}^{\infty} (-1)^q \dim H^q(X, \mathcal{F}).$$

   We showed that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of coherent sheaves then

   $$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

Using the formulas from Lecture 16, we calculated $\chi(\mathcal{O}_{\mathbb{P}^n}(d))$ explicitly as

$$\chi(\mathcal{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{d} = \frac{(d+1)(d+2)\dots(d+n)}{n!}.$$

We deduced from this and from the syzygy theorem that $\chi(\mathcal{F}(d))$ is a polynomial in d (for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$). Here $\mathcal{F}(d) := \mathcal{F} \otimes \mathcal{O}(d)$.

3. By adapting the proof of the finite dimensionality in (1) (namely, descending induction on q) we showed that for a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, the groups $H^q(\mathbb{P}^n, \mathcal{F}(d))$ are zero for $q > 0$ and $d \gg 0$.
4. We defined Ω^q_X, $\mathcal{T}_X$, and ω_X (see Definition 17.3.1) and covered the rest of §17.3.