16. Lecture 16 (Apr 2): Cohomology of coherent sheaves on projective varieties

Recommended reading: Hartshorne III.4 and III.5, Kempf §9.1–9.3

16.1. Čech cohomology

We shall now use the vanishing of cohomology of quasi-coherent sheaves on affines to give an algorithm for computing cohomology on an arbitrary separated algebraic set.

Let X be a separated algebraic set (or noetherian scheme), $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module, and $U \subseteq X$ an affine open subset. Denote by $j: U \to X$ the inclusion map, which is an affine morphism: the preimage $j^{-1}(V) = U \cap V$ of every affine open $V \subseteq X$ is affine. Consequently, the maps

$$j^*: H^q(X, j_*(\mathcal{F}|_U)) \longrightarrow H^q(U, \mathcal{F}|_U)$$

are isomorphisms, and hence $H^q(X, j_*(\mathcal{F}|_U)) = 0$ for $q > 0$. The restriction maps $\mathcal{F}(V) \to \mathcal{F}(U \cap V)$ for varying $V \subseteq X$ assemble to give a canonical "restriction" map $\mathcal{F} \to j_*(\mathcal{F}|_U)$ towards a sheaf with no higher cohomology. This observation is the basis for the construction of the Čech complex of $\mathcal{F}$.

Construction. Let $X = U_0 \cup \cdots \cup U_r$ be a finite affine open cover. For a non-empty $I \subseteq \{0, \ldots, r\}$ denote by $U_I = \bigcap_{i \in I} U_i$ the corresponding intersection (which again is affine since X is separated), by $j_I: U_I \to X$ the inclusion map, and let

$$\mathcal{F}_I = j_{I,*}(\mathcal{F}|_{U_I}) = j_{I,*} j_I^* \mathcal{F}.$$

Consider the following complex of quasi-coherent sheaves on X:

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{|I|=1} \mathcal{F}_I \longrightarrow \bigoplus_{|I|=2} \mathcal{F}_I \longrightarrow \ldots \qquad (16.1.1)$$

where the map $\mathcal{F} \to \bigoplus_{|I|=1} \mathcal{F}_I$ is the sum of the restriction maps $\mathcal{F} \to \mathcal{F}_I$, and where the differential

$$d: \bigoplus_{|I|=p} \mathcal{F}_I \longrightarrow \bigoplus_{|J|=p+1} \mathcal{F}_I$$

is given by

$$d(f_I)_J = \sum_{j \in J} (-1)^{s(j)} f_{J \setminus \{j\}}.$$

Here $s(j) = \#\{j' \in J : j' < j\}$, so if $J = \{j_0 < \cdots < j_p\}$ then $s(j_m) = m$.

Lemma 16.1.1. *The complex of sheaves* (16.1.1) *is exact.*

Proof. Not to obscure a simple picture with overwhelming notation, consider first the case $r = 1$, so $X = U_0 \cap U_1$. Let $x \in X$, it suffices to show exactness on stalks at x. Without loss of generality, we may assume that $x \in U_0$. In this case $(\mathcal{F}_0)_x = \mathcal{F}_x$ and a moment's thought shows that $(\mathcal{F}_{01})_x = (\mathcal{F}_1)_x$ (as U_1 and U_{01} are equal in a neighborhood of x, namely on U_0). The stalks of the Čech complex at x thus takes the form

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}_x \oplus (\mathcal{F}_1)_x \longrightarrow (\mathcal{F}_1)_x \longrightarrow 0.$$

This is visibly (split) exact.

For the general case, let again $x \in X$, and assume $x \in U_0$. Then for every $I \subseteq \{0, \ldots, r\}$ the opens U_I and $U_{I \cup \{0\}\}}$ are equal in a neighborhood of x, and hence

$$(\mathcal{F}_I)_x = (\mathcal{F}_{I \cup \{0\}})_x.$$

If we set $C_p = \bigoplus_{I\subseteq\{1,\dots,r\},|I|=p+1}(\mathcal{F}_I)_x$, then we can decompose the stalk at x of the p-th term of the Čech complex of $\mathcal{F}$ as follows

$$\bigoplus_{|I|=p+1}(\mathcal{F}_I)_x = \bigoplus_{0\in I}(\mathcal{F}_I)_x \oplus \bigoplus_{I\subseteq\{1,\dots,r\}}(\mathcal{F}_I)_x = C_p \oplus C_{p-1}.$$

The stalk of the complex (16.1.1) at x takes the following form

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}_x \oplus C_0 \longrightarrow C_0 \oplus C_1 \longrightarrow C_1 \oplus C_2 \longrightarrow \dots$$

which again is split exact. □

Definition 16.1.2. Let X be a separated algebraic set (or noetherian scheme), let $X = U_0 \cup \dots \cup U_r$ be a finite affine open cover, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The **Čech complex** of $\mathcal{F}$ relative to the cover $\{U_i\}_{i=0}^r$ is the complex

$$\mathcal{C}^\bullet(\mathcal{F},\{U_i\}) = \Big[\ \oplus_{|I|=1}\mathcal{F}(U_I) \longrightarrow \oplus_{|I|=2}\mathcal{F}(U_I) \longrightarrow \dots\ \Big]$$

obtained by taking the global sections of the complex (16.1.1).

Corollary 16.1.3. *Let X be a separated algebraic set (or noetherian scheme) and let $X = U_0 \cup \dots \cup U_r$ be a finite affine open cover. Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ on X we have*

$$H^q(X,\mathcal{F}) = H^q(\mathcal{C}^\bullet(\mathcal{F},\{U_i\})).$$

Proof. Since each U_I is affine, we have $H^q(X,\oplus_{|I|=p}\mathcal{F}_I) = 0$ for $q > 0$ and all $p \geq 0$ (see the paragraph preceding our construction of the Čech complex). We conclude applying Lemma 16.1.1 and Lecture 15, Lemma 15.2.2(c). □

Corollary 16.1.4. *Let X be a separated scheme which can be covered by $r+1$ affine open subsets. Then $H^q(X,\mathcal{F}) = 0$ for $q > r$ and every quasi-coherent sheaf $\mathcal{F}$ on X.*

Proof. This follows from Corollary 16.1.3 since $\mathcal{C}^\bullet(\mathcal{F},\{U_i\})$ is concentrated in degrees $[0,r]$. □

Example 16.1.5 (Punctured plane)**.** Let $X = \mathbb{A}^2 \setminus \{(0,0)\}$ with coordinates x_0, x_1, covered by $U_i = D(x_i)$. Let us compute $H^q(X,\mathcal{O}_X)$ using the associated Čech complex

$$\begin{array}{ccc} [\mathcal{O}(U_0)\oplus\mathcal{O}(U_1) & \xrightarrow{\quad d \quad} & \mathcal{O}(U_{01})] \\ \| & & \| \\ k[x_0^{\pm1},x_1]\oplus k[x_0,x_1^{\pm1}] & \xrightarrow[(f_0,f_1)\mapsto f_0-f_1]{} & k[x_0^{\pm1},x_1^{\pm1}]. \end{array}$$

The kernel of the bottom map is $k[x_0,x_1]$ embedded diagonally (and we knew already that $\mathcal{O}(X) = k[x_0,x_1]$), and the cokernel is spanned by monomials $x_0^{n_0}x_1^{n_1}$ with both exponents strictly negative, which we can write as

$$H^1(X,\mathcal{O}_X) = x_0^{-1}x_1^{-1}k[x_0^{-1},x_1^{-1}].$$

The groups $H^q(X,\mathcal{O}_X)$ are zero for $q > 1$.

We mention the following result without proof.

Theorem 16.1.6 (Grothendieck vanishing, [Hartshorne III 2.7])**.** *Let X be a noetherian topological space of dimension n. Then $H^q(X,\mathcal{F}) = 0$ for $q > n$ and every abelian sheaf $\mathcal{F}$ on X.*

16.2. Cohomology of projective space

Our next goal is to compute the groups

$$H^q(\mathbb{P}^n, \mathcal{O}(d))$$

for all $q \geq 0$, $n \geq 1$, and $d \in \mathbb{Z}$. As we shall see later (Theorem **??**), every coherent sheaf on $\mathbb{P}^n$ is a quotient of a finite direct sum of some $\mathcal{O}(d_i)$'s, and so our computation will have some fairly general consequences. We start our calculation with the structure sheaf on $\mathbb{A}^{n+1} \setminus \{0\}$, which is only a bit harder than the one for $n = 1$ done in Example 16.1.5.

Lemma 16.2.1. *For $n \geq 1$ we have*

$$H^q(\mathbb{A}^{n+1} \setminus \{0\}, \mathcal{O}) = \begin{cases} k[x_0, \dots, x_n] & q = 0 \\ 0 & 0 < q < n \\ (x_0 \dots x_n)^{-1} k[x_0^{-1}, \dots, x_n^{-1}] & q = n \\ 0 & q > n. \end{cases}$$

Proof. The proof is by induction on n. We have shown this for $n = 1$ in Example 16.1.5. Suppose that $n > 1$ and that the result holds for $n - 1$. Let $U = U_n = D(x_n)$ and let $j: U \to X$ be the inclusion. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow j_* \mathcal{O}_U \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{16.2.1}$$

where $\mathcal{Q}$ is supported on the closed subset $X' = V(x_n) \simeq \mathbb{A}^n \setminus \{0\}$. Even though it is not a $\mathcal{O}_{X'}$-module, it decomposes as a direct sum of $\mathcal{O}_{X'}$-modules

$$\mathcal{Q} = \bigoplus_{m<0} \mathcal{O}_{X'} \cdot x_n^m.$$

Since X' is noetherian, cohomology commutes with infinite direct sums, so by the induction assumption we obtain

$$H^q(X, \mathcal{Q}) = H^q(X', \mathcal{Q}) = \begin{cases} x_n^{-1} k[x_0, \dots, x_{n-1}][x_n^{-1}] & q = 0 \\ 0 & 0 < q < n-1 \\ (x_0 \dots x_n)^{-1} k[x_0^{-1}, \dots, x_{n-1}^{-1}, x_n^{-1}] & q = n-1 \\ 0 & q > n-1. \end{cases}$$

Since U is affine, $H^q(X, j_* \mathcal{O}_U) = H^q(U, \mathcal{O}_U) = 0$. Thus the long cohomology exact sequence associated to (16.2.1) gives

$$H^1(X, \mathcal{O}_X) = \operatorname{cok}(\Gamma(U, \mathcal{O}_U) \to \Gamma(X, \mathcal{Q})) = \operatorname{cok}(k[x_0, \dots, x_n, x_n^{-1}] \to x_n^{-1} k[x_0, \dots, x_n^{-1}]),$$

which is zero, and $H^q(X, \mathcal{O}_X) \simeq \oplus_{m<0} H^{q-1}(X', \mathcal{O}'_X) \cdot x_n^m$. Thus we conclude by induction. □

Lemma 16.2.2. *Let $\pi: \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the projection. Then*

$$\pi_* \mathcal{O} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d).$$

Proof. We already know that for an open $U \subseteq \mathbb{P}^n$, we have $\mathcal{O}(d)(U) = \mathcal{O}(\pi^{-1}(U))_d$, the space of all functions on $\pi^{-1}(U) \subseteq \mathbb{A}^{n+1}$ which are homogeneous of degree d. So we need to show that every function on $\pi^{-1}(U) \subseteq \mathbb{A}^{n+1}$ can be written (uniquely) as a sum of homogeneous functions. Uniqueness is checked easily, and then existence can be checked locally. We can therefore reduce to the case $U = D(g)$ for some homogeneous g, in which case $\mathcal{O}(U) = k[x_0, \dots, x_n][g^{-1}]$. An element of this ring is a unique sum of f/g^m for homogeneous f of degree e, and such an element gives a homogeneous function of degree $e - md$. □

Theorem 16.2.3. *Let $n \geq 1$, $d \in \mathbb{Z}$, $q \geq 0$. Then*

$$H^q(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} k[x_0,\dots,x_n]_d & q = 0 \\ 0 & 0 < q < n \\ \left((x_0 \dots x_n)^{-1} k[x_0^{-1},\dots,x_n^{-1}]\right)_d \simeq (x_0\dots x_n)^{-1}k[x_0^{-1},\dots,x_n^{-1}]_{d+n+1} & q = n \\ 0 & q > n. \end{cases}$$

Proof. Since the map $\pi: \mathbb{A}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$ is affine, by Lemma 16.2.2 we have

$$H^q(\mathbb{A}^{n+1} \smallsetminus \{0\}, \mathcal{O}) = H^q(\mathbb{P}^n, \pi_* \mathcal{O}) = H^q(\mathbb{P}^n, \oplus_{d\in\mathbb{Z}} \mathcal{O}(d)) = \bigoplus_{d\in\mathbb{Z}} H^q(\mathbb{P}^n, \mathcal{O}(d)).$$

On the other hand, the left-hand side is given in Lemma 16.2.1. Moreover, for $q = 0$ the degree d part of the right-hand side corresponds to $k[x_0,\dots,x_n]_d$ on the left (this is again Lemma 16.2.2). For $q = n$, a closer inspection of the Čech complex computation of $H^n(\mathbb{A}^{n+1} \smallsetminus \{0\}, \mathcal{O})$ (which we omit) shows that the degree d part of the right-hand side corresponds to the degree d part of $(x_0\dots x_n)^{-1}k[x_0^{-1},\dots,x_n^{-1}]$. $\square$

Corollary 16.2.4. *The spaces $H^q(\mathbb{P}^n, \mathcal{O}(d))$ are finite-dimensional. Their dimensions are given as*

$$H^q(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} \binom{n+d}{n} & q = 0, d \geq 0 \\ 0 & q = 0, d < 0 \\ 0 & 0 < q < n \\ \binom{-d-1}{n} & q = n, d < 0 \\ 0 & q > n. \end{cases}$$

The following result is one of the most important tools. We omit the proof (it is not so difficult given what we know).

Theorem 16.2.5 (Serre, [Hartshorne II 5.17 and 5.18])**.** *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Then there exists an integer d, an $N \geq 0$, and a surjection*

$$\mathcal{O}(-d)^N \longrightarrow \mathcal{F}.$$

Remark 16.2.6. The following equivalent form of Theorem 16.2.5 is often given. Let us call $\mathcal{F}$ **globally generated** if the natural map

$$\Gamma(\mathbb{P}^n, \mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{F}$$

is surjective. (Note for example that if $\mathcal{F}$ is an invertible sheaf this simply means: for every $x \in \mathbb{P}^n$ there exists a global section of $\mathcal{F}$ which does not vanish at x.) Then Theorem 16.2.5 is equivalent to the statement that for $d \gg 0$, the coherent sheaf $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}(d)$ is globally generated.

Corollary 16.2.7. *Let X be a projective algebraic set and let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Then the cohomology groups $H^q(X, \mathcal{F})$ are finite-dimensional for all $q \geq 0$. In particular, $\Gamma(X, \mathcal{F})$ is finite-dimensional.*

Proof. Let $i: X \to \mathbb{P}^n$ be a closed immersion into some projective space. Since i is finite, the sheaf $i_*\mathcal{F}$ is coherent and $H^q(X, \mathcal{F}) \simeq H^q(\mathbb{P}^n, i_*\mathcal{F})$. Thus we may assume that $X = \mathbb{P}^n$.

We prove the result by *descending* induction on q. For $q > n$ we have $H^q(\mathbb{P}^n, \mathcal{F}) = 0$ since $\mathbb{P}^n$ can be covered by $n + 1$ affines. For the induction step, let $q > 0$ and suppose the result has been proved for all

$q' > q$. By Theorem 16.2.5 we have a surjection $\mathcal{O}(-d)^N \longrightarrow \mathcal{F}$, and letting $\mathcal{R}$ be its kernel, a short exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{O}(-d)^N \longrightarrow \mathcal{F} \longrightarrow 0.$$

The relevant portion of the long cohomology exact sequence is

$$H^q(\mathbb{P}^n, \mathcal{O}(-d))^N \longrightarrow H^q(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^{q+1}(\mathbb{P}^n, \mathcal{R})$$

where the group on the left is finite-dimensional by Corollary 16.2.4 and the one on the right is finite-dimensional by induction assumption. Thus $H^q(X, \mathcal{F})$ is finite-dimensional. □