

15. Lecture 15 (Mar 19): Cohomology

Recommended reading: Kempf §8, Hartshorne III §2–3

15.1. Motivation

Let X be an algebraic set (or a noetherian scheme) and \mathcal{F} a coherent sheaf on X . In this lecture, we are going to associate to \mathcal{F} a sequence of vector spaces $H^q(X, \mathcal{F})$ (indexed by $q \geq 0$) called its *cohomology groups*. In brief, for $q = 0$, these are the global sections $\Gamma(X, \mathcal{F})$, and for a short exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$, the group $H^1(X, \mathcal{F}')$ allows us to control the failure of it staying exact after applying H^0 , then $H^2(X, \mathcal{F}')$ addresses a similar problem with H^1 , and so on.

Surprisingly, the actual definition of cohomology groups is almost unimportant, and it is challenging to give elements of $H^q(X, \mathcal{F})$ a concrete geometric meaning for $q \geq 2$ (for $q = 1$, see Remark 15.2.7). What *is* important is how to compute them, and how to use their properties to prove new results in algebraic geometry.

What are the key properties of cohomology?

1. A short exact sequences of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

induces long exact sequences of cohomology groups

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(X, \mathcal{F}') & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{F}'') \\
 & & & & & & \downarrow \\
 & & & & & & H^1(X, \mathcal{F}') \longrightarrow \dots \\
 & & & & & & \downarrow \\
 & & & & & & \dots \longrightarrow H^{q-1}(X, \mathcal{F}'') \\
 & & & & & & \downarrow \\
 & & & & & & H^q(X, \mathcal{F}') \longrightarrow H^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F}'') \\
 & & & & & & \downarrow \\
 & & & & & & H^{q+1}(X, \mathcal{F}') \longrightarrow \dots
 \end{array}$$

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2. If X is affine, then $H^q(X, \mathcal{F}) = 0$ for $q > 0$ (Theorem 15.3.1).
3. In general (assuming X is separated), $H^q(X, \mathcal{F})$ can be computed using the Čech complex associated to a fixed affine open cover of X .
4. We have $H^q(X, \mathcal{F}) = 0$ if X can be covered by q affine open subsets, or if $\dim(X) < q$.
5. If X is projective (or more generally complete), then $H^q(X, \mathcal{F})$ are finite-dimensional for all $q \geq 0$.

Up to 70% of core technical results in algebraic geometry rely on the computation or vanishing of some $H^q(X, \mathcal{O}_X(D))$. Here are two basic but important examples, a preview of things to come.

Example 15.1.1 (Connectedness of hyperplane sections). Let $X \subseteq \mathbb{P}^n$ be a normal projective variety of dimension ≥ 2 and let $D = X \cap H$ for a hyperplane $H \subseteq \mathbb{P}^n$. Then D is connected.

where $\omega_X = \Omega_X^1$ is the **canonical invertible sheaf** (the sheaf of Kähler differentials, or of algebraic differential one-forms on X). In particular, if $\deg(D) > \deg(\omega_X)$, then $\text{Hom}(\mathcal{O}_X(D), \omega_X) = 0$, and consequently $H^1(X, \mathcal{O}_X(D)) = 0$. (As we shall see later as a consequence of the Riemann–Roch theorem, $\deg(\omega_X) = 2g - 2$ where

$$g = \dim H^0(X, \omega_X) = \dim H^1(X, \mathcal{O}_X)$$

is the **genus** of X .) We conclude that for $n \gg 0$ (more precisely, for $n \geq 2g$) there exists a rational function f with pole of order exactly n at P and regular on U .

We can draw from this the following important corollary: the curve $U = X \setminus \{P\}$ is affine (and more generally, every proper open subset of X is affine). Indeed, reasoning as in the last problem session (see Lecture 14, §14.5) we see that the function f we have just constructed corresponds to a finite map

$$f: X \longrightarrow \mathbb{P}^1$$

with $f^{-1}(\infty) = \{P\}$. Thus $U = f^{-1}(\mathbb{P}^1 \setminus \{\infty\}) = f^{-1}(\mathbb{A}^1)$ is affine.

15.2. Definition and basic properties of sheaf cohomology

The following elementary exposition of sheaf cohomology follows Kempf §8.1.

Let \mathcal{F} be an abelian sheaf on a topological space X . In our discussion of sheafification in Lecture 7, we introduced the **sheaf of discontinuous sections**

$$D(\mathcal{F}) = \prod_{x \in X} i_{x,*} \mathcal{F}_x,$$

the product of skyscraper sheaves corresponding to the stalks of \mathcal{F} at all points of x . We have a tautological injective map

$$i: \mathcal{F} \longrightarrow D(\mathcal{F}).$$

The sheaf $D(\mathcal{F})$ is an example of a **flabby** (a.k.a. **flasque**) sheaf: one for which the restriction maps $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ are surjective for all $V \subseteq U \subseteq X$. We recall the key property of flabby sheaves (see Problem Set 3):

- If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is a short exact sequence with \mathcal{F}' flabby, then

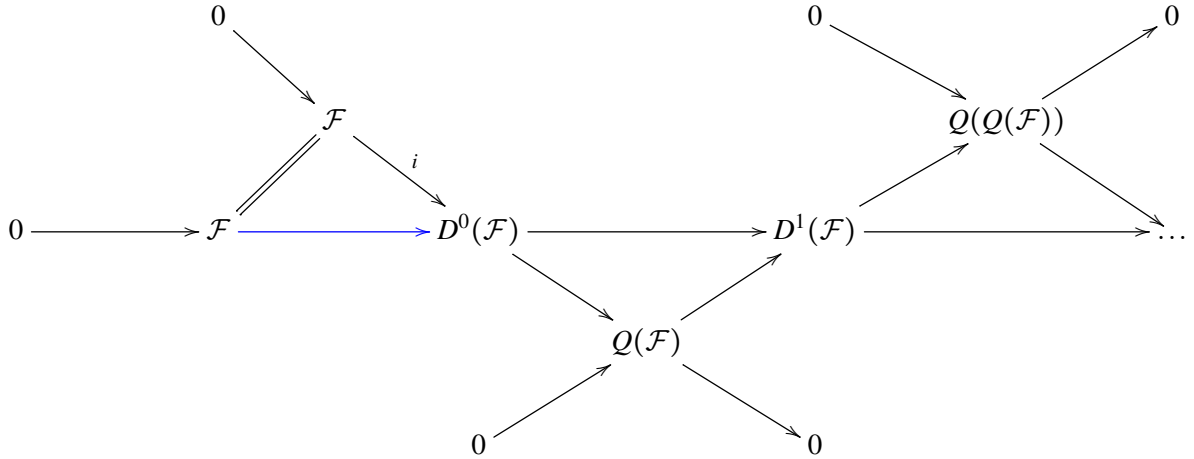
$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \longrightarrow 0$$

is exact.

Using the construction $D(\mathcal{F})$ iteratively, we can associate to every abelian sheaf \mathcal{F} a canonical **flabby resolution** $D^\bullet(\mathcal{F})$ (a.k.a. the *Godement resolution*): define $Q(\mathcal{F}) = D(\mathcal{F})/\mathcal{F}$ and set

$$D^0(\mathcal{F}) = D(\mathcal{F}), \quad D^1(\mathcal{F}) = D(Q(\mathcal{F})), \quad D^2(\mathcal{F}) = D(Q(Q(\mathcal{F}))), \quad \dots$$

and so on. Diagram for visual aid:



This being a resolution means that the middle row is exact. Applying the functor $\Gamma(X, -)$ to the complex of abelian sheaves $D^\bullet(\mathcal{F})$ we obtain a complex of abelian groups $\Gamma(X, D^\bullet(\mathcal{F}))$.

Definition 15.2.1. Let \mathcal{F} be an abelian sheaf on a topological space X . For $q \geq 0$, the q -th cohomology group of \mathcal{F} is

$$H^q(X, \mathcal{F}) = H^q(\Gamma(X, D^\bullet(\mathcal{F}))) = \frac{\ker(\Gamma(X, D^q(\mathcal{F})) \rightarrow \Gamma(X, D^{q+1}(\mathcal{F})))}{\text{im}(\Gamma(X, D^{q-1}(\mathcal{F})) \rightarrow \Gamma(X, D^q(\mathcal{F})))}.$$

We observe right away that left-exactness of $\Gamma(X, -)$ applied to $0 \rightarrow \mathcal{F} \rightarrow D^0(\mathcal{F}) \rightarrow D^1(\mathcal{F})$ gives

$$\Gamma(X, \mathcal{F}) \xrightarrow{\sim} H^0(X, \mathcal{F}).$$

Note first of all that since $D(\mathcal{F})$ is obviously functorial in \mathcal{F} , so are all of the above constructions, in particular $H^q(X, \mathcal{F})$. Furthermore, if

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is a short exact sequence of abelian sheaves, then so are

$$0 \longrightarrow D(\mathcal{F}') \longrightarrow D(\mathcal{F}) \longrightarrow D(\mathcal{F}'') \longrightarrow 0$$

and therefore also

$$0 \longrightarrow Q(\mathcal{F}') \longrightarrow Q(\mathcal{F}) \longrightarrow Q(\mathcal{F}'') \longrightarrow 0.$$

Iterating this, it follows that the sequence of complexes of sheaves

$$0 \longrightarrow D^\bullet(\mathcal{F}') \longrightarrow D^\bullet(\mathcal{F}) \longrightarrow D^\bullet(\mathcal{F}'') \longrightarrow 0$$

is exact. By our “key property” of flabby sheaves, it remains exact after applying $\Gamma(X, -)$, so we have a short exact sequence of complexes of abelian groups

$$0 \longrightarrow \Gamma(X, D^\bullet(\mathcal{F}')) \longrightarrow \Gamma(X, D^\bullet(\mathcal{F})) \longrightarrow \Gamma(X, D^\bullet(\mathcal{F}'')) \longrightarrow 0.$$

By the snake lemma, we obtain a long exact sequence of cohomology groups

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(X, \mathcal{F}') & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{F}'') \\
& & \dots & \longrightarrow & H^{q-1}(X, \mathcal{F}') & \longrightarrow & H^{q-1}(X, \mathcal{F}) & \longrightarrow & H^{q-1}(X, \mathcal{F}'') \\
& & & & \searrow & & \searrow & & \searrow \\
& & & & & & & & \delta \\
& & & & & & & & \delta \\
& & & & & & & & \dots
\end{array}$$

Lemma 15.2.2. (a) If \mathcal{F} is flabby then $H^q(X, \mathcal{F}) = 0$ for $q > 0$.

(b) If $\mathcal{F} \rightarrow \mathcal{C}^\bullet$ is a flabby resolution (meaning that $0 \rightarrow \mathcal{F} \rightarrow \mathcal{C}^0 \rightarrow \mathcal{C}^1 \rightarrow \dots$ is exact and all the \mathcal{C}^i are flabby), then we have a canonical isomorphism

$$H^q(X, \mathcal{F}) = H^q(\Gamma(X, \mathcal{C}^\bullet)).$$

(c) More generally, (b) holds under the weaker (thanks to (a)) assumption that

$$H^q(X, \mathcal{C}^p) = 0 \quad \text{for } q > 0 \text{ and all } p \geq 0.$$

Proof. (a) We first check the following auxiliary fact: if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is a short exact sequence with \mathcal{F}' and \mathcal{F} flabby, then \mathcal{F}'' is flabby as well. To show this, take $V \subseteq U \subseteq X$ and consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F}'(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}''(U) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}'(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{F}''(V) & \longrightarrow & 0.
\end{array}$$

Here the rows are exact since \mathcal{F}' is flabby (by our “key fact” applied to the restriction of the sequence to U or V), and the left and middle vertical maps are surjective since \mathcal{F}' and \mathcal{F} are flabby. It follows that the right map is surjective (by the snake lemma or an easy diagram chase), and since U and V were arbitrary, we see that \mathcal{F}'' is flabby.

Now, the auxiliary fact implies that $Q(\mathcal{F})$ is flabby. By induction, all $Q^i(\mathcal{F})$ are flabby, and the “key fact” implies that our big “visual aid” diagram remains exact after applying $\Gamma(X, -)$. Thus $\Gamma(X, D^\bullet(\mathcal{F}))$ is exact in positive degrees, and hence $H^q(X, \mathcal{F}) = 0$ for $q > 0$.

(c) Induction on $q \geq 0$. This is true for $q = 0$ (with no assumptions on the \mathcal{C}^p). Let \mathcal{G} be the cokernel of $\mathcal{F} \rightarrow \mathcal{C}^0$ so that $\mathcal{G} \rightarrow \mathcal{C}^{\bullet \geq 1}$ gives a resolution of \mathcal{G} of the same kind, and the cohomology sequence of the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{G} \longrightarrow 0$$

gives

$$H^1(X, \mathcal{F}) = \text{cok}(H^0(X, \mathcal{C}^0) \rightarrow H^0(X, \mathcal{G})) = H^1(\Gamma(X, \mathcal{C}^\bullet)), \quad H^q(X, \mathcal{F}) \simeq H^{q-1}(X, \mathcal{G}) \quad (q \geq 2),$$

so the result for $(\mathcal{G}, q-1)$ implies the result for (\mathcal{F}, q) . \square

Lemma 15.2.3. Let X be a noetherian topological space and let $\{\mathcal{F}_\alpha\}$ be an inductive system of abelian sheaves on X , and let $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$ be its direct limit. Then

$$\varinjlim H^q(X, \mathcal{F}_\alpha) \xrightarrow{\sim} H^q(X, \mathcal{F})$$

for all $q \geq 0$.

Proof. We have (hopefully) shown this for $q = 0$. The general case is left as a (worthwhile) exercise. \square

Remark 15.2.4. More generally, the result holds on any spectral space (we do not need *all* opens to be quasi-compact, but only a basis of quasi-compact opens, stable under intersections).

Exercise 15.2.5. Compute $H^1(S^1, \mathbb{Z})$ directly from our definition of cohomology.

Remark 15.2.6 (Functoriality). Trivially, $H^q(X, \mathcal{F})$ is functorial in \mathcal{F} . It is also functorial in X , in the following way(s). Let $f: Y \rightarrow X$ be a map and let \mathcal{F} be an abelian sheaf on Y . Then we have natural maps

$$f^*: H^q(X, \pi_* \mathcal{F}) \longrightarrow H^q(Y, \mathcal{F})$$

defined as follows (more abstract definitions of cohomology yield easier constructions). For every flabby sheaf \mathcal{G} on Y , the push-forward $f_* \mathcal{G}$ is flabby. It follows that we have a complex on X

$$0 \longrightarrow \pi_* \mathcal{F} \longrightarrow \pi_* D^0(\mathcal{F}) \longrightarrow \pi_* D^1(\mathcal{F}) \longrightarrow \dots$$

whose terms $\pi_* D^p(\mathcal{F})$ are flabby for $p > 0$, and which is exact on the left (but not exact in general). Let

$$\mathcal{Q}^p = \text{im}(\pi_* D^{p-1}(\mathcal{F}) \rightarrow \pi_* D^p(\mathcal{F})) = \text{cok}(\pi^* \mathcal{Q}^{p-1}(\mathcal{F}) \rightarrow \pi_* D^{p-1}(\mathcal{F})).$$

We have natural maps $H^q(X, \mathcal{Q}^p) \rightarrow$

Here is the concrete interpretation of H^1 promised in the introduction.

Remark 15.2.7 (H^1 and torsors). Let \mathcal{G} be a sheaf of groups (not necessarily commutative) on a topological space X . A \mathcal{G} -torsor on X is a sheaf of sets \mathcal{F} on X together with a map $\mathcal{F} \times \mathcal{G} \rightarrow \mathcal{F}$ satisfying the following axioms:

- For every open $U \subseteq X$, the map $\mathcal{F}(U) \times \mathcal{G}(U) \rightarrow \mathcal{F}(U)$ defines a (right) action of the group $\mathcal{G}(U)$ on the set $\mathcal{F}(U)$.
- For every open $U \subseteq X$ such that $\mathcal{F}(U) \neq \emptyset$, the above action is free and transitive.
- There exists an open cover $X = \bigcup U_\alpha$ such that $\mathcal{F}(U_\alpha) \neq \emptyset$.

For example $\mathcal{F} = \mathcal{G}$ with its natural \mathcal{G} -action by right multiplication is a \mathcal{G} -torsor, called the **trivial torsor**. A morphism between \mathcal{G} -torsors is a \mathcal{G} -equivariant map $\mathcal{F} \rightarrow \mathcal{F}'$. It is easy to see that such a map is necessarily an isomorphism. Moreover, maps $\mathcal{G} \rightarrow \mathcal{F}$ from the trivial \mathcal{G} -torsor to a \mathcal{G} -torsor \mathcal{F} are in bijection with $H^0(X, \mathcal{F})$. Thus \mathcal{F} is trivial if and only if $\mathcal{F}(X) \neq \emptyset$.

Suppose now that \mathcal{G} is an abelian sheaf. To a \mathcal{G} -torsor \mathcal{F} we can associate a class $c(\mathcal{F}) \in H^1(X, \mathcal{G})$ as follows.

15.3. Cohomology of an affine

Theorem 15.3.1. *Let \mathcal{F} be a quasi-coherent sheaf on an affine algebraic set¹ X . Then*

$$H^q(X, \mathcal{F}) = 0 \quad \text{for } q > 0.$$

Ideally, this should be deduced from the exactness of the Čech complex (Lecture 9, §9.1). Grothendieck does this in EGA III, using comparison with Čech cohomology based on a theorem of Cartan (Godement *Geometrie algébrique et théorie des faisceaux*, II 5.9.2). Since we want to avoid a lengthy discussion of Čech cohomology, we give a different proof.

Proof. To be added later. See [Kempf §8.2] or [Hartshorne Theorem III 3.5]. □

Remark 15.3.2. Theorem 15.3.1 allows us to “rediscover” the fact (proved in Lecture 11) that if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is a short exact sequence of sheaves on an affine X with \mathcal{F}' a quasi-coherent \mathcal{O}_X -module, then

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \longrightarrow 0$$

is exact. Indeed, this is a consequence of the vanishing $H^1(X, \mathcal{F}') = 0$.

Remark 15.3.3. Serre proved the following converse to Theorem 15.3.1 (see [Hartshorne Theorem III 3.7]): suppose that X is an algebraic set or an affine noetherian scheme such that $H^1(X, \mathcal{I}) = 0$ for every coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. Then X is affine.

Sketch of proof: The assumption implies that for every closed subscheme/subset $Z \subseteq X$, the restriction $\mathcal{O}(X) \rightarrow \mathcal{O}(Z)$ is surjective. If $U \subseteq X$ is an open subset and $x \in U$, applying this to $Z = (X \setminus U) \cup \{x\}$ and the function $\chi_{\{x\}}$ (1 on $\{x\}$, 0 on $X \setminus U$) we find an $f \in \mathcal{O}(X)$ such that $x \in D(f) \subseteq U$. Thus $D(f)$'s form a base of the topology on X . Let $X = \bigcup U_\alpha$ be an affine open cover. We may find $f_1, \dots, f_r \in \mathcal{O}(X)$ (finitely many since X is qc) such that $X = D(f_1) \cup \dots \cup D(f_r)$ and for every i there exists an α such that $D(f_i) \subseteq U_\alpha$. Now $D(f_i) = \text{Spec}(\mathcal{O}(U_\alpha)[f_i^{-1}])$ are affine. Thus, by the (corrected) Problem 3 on Problem Set 4, to show that X is affine it suffices to prove that f_1, \dots, f_r generate the unit ideal in $\mathcal{O}(X)$. To this end, we look at the short exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{O}_X^r \xrightarrow{(f_1, \dots, f_r)} \mathcal{O}_X \longrightarrow 0;$$

the map on the right is surjective since the $D(f_i)$ cover X , so that on $D(f_i)$ the sequence has a splitting, and \mathcal{R} (the “sheaf of relations” between the f_i) is defined as the kernel. If $H^1(X, \mathcal{R}) = 0$, the map $(f_1, \dots, f_r): \mathcal{O}(X)^r \rightarrow \mathcal{O}(X)$ is surjective, so $1 = \sum g_i f_i$ for some $g_i \in \mathcal{O}(X)$ and we win.

To show that $H^1(X, \mathcal{R}) = 0$ we define $\mathcal{R}_i = \mathcal{R} \cap \mathcal{O}_X^i$ for $i = 0, \dots, r$. We have $\mathcal{R} = \mathcal{R}_r$, $\mathcal{R}_{i-1} \subseteq \mathcal{R}_i$, and the quotients $\mathcal{R}_i/\mathcal{R}_{i-1}$ embed into \mathcal{O}_X , so that $H^1(X, \mathcal{R}_i/\mathcal{R}_{i-1}) = 0$. The cohomology sequences associated to the short exact sequences

$$0 \longrightarrow \mathcal{R}_{i-1} \longrightarrow \mathcal{R}_i \longrightarrow \mathcal{R}_i/\mathcal{R}_{i-1} \longrightarrow 0$$

allow us to show by induction on i that $H^1(X, \mathcal{R}_i) = 0$ for all i .

Corollary 15.3.4. *Let $f: Y \rightarrow X$ be an affine map and let \mathcal{F} be a quasi-coherent sheaf on Y . Then the maps*

$$f^*: H^q(X, f_* \mathcal{F}) \longrightarrow H^q(Y, \mathcal{F})$$

are isomorphisms for all $q \geq 0$.

¹The same proof works for a noetherian scheme. The result is true for non-noetherian schemes as well.

Example 15.3.5. Examples of affine maps to keep in mind:

- (a) closed immersions;
- (b) the inclusion of an affine open $j:U \hookrightarrow X$ into a separated X ;
- (c) the projection $\pi:\mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$.