13. Lecture 13 (Mar 12): Invertible sheaves on affine and projective spaces

Recommended reading: Vakil 15.1

13.1. Preview: the Riemann hypothesis for curves over finite fields

We start with a discussion (unrelated to where we are in the course) of point counting problems over fields.

Basic facts about finite fields:

- for every prime p and every $e \geq 1$ there exists a (unique up to isomorphism) field $\mathbb{F}_q$ with $q = p^e$ elements;

- finite extensions of $\mathbb{F}_q$ are of the form $\mathbb{F}_{q^n}$ for some $n \geq 1$ (it is the splitting field of the separable polynomial $x^{q^n} - x$).

Let $k = \mathbb{F}_q$ be a finite field with $q = p^e$ elements, let $\bar{k}$ be a fixed algebraic closure of k, and let X be a projective variety over k. Since we have not discussed varieties over non-algebraically closed field, let us simply say that we have a projective variety X over $\bar{k}$ which is the zero set in $\mathbb{P}^n$ of a homogeneous ideal $I \subseteq \bar{k}[x_0, \ldots, x_n]$ which is generated by elements of $k[x_0, \ldots, x_n]$ (in simple words, X is defined in $\mathbb{P}^n_{\bar{k}}$ by a system of homogeneous equations with coefficients in the finite field k). By $X(k)$ we denote the set of points of X with coordinates in k (called the set of k-rational points of X). It is a finite set, and we denote its cardinality by $\#X(k)$.

We will be interested in the case when X is a smooth curve. Its **genus** $g = g(X)$ is a non-negative integer which will be defined later (as the dimension of the space $\Gamma(X, \Omega^1_X)$ of differentials on X), for now it is enough to know that if X is a plane curve, cut out of $\mathbb{P}^2$ by a homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d, then

$$g(X) = \frac{(d-1)(d-2)}{2}.$$

Thus, the genus of a line, conic, cubic, quartic is 0, 0, 1, 3. (Since there exist curves of every genus $g \geq 0$, as we shall see later, it follows from the asymptotics of the above formula that most curves do not embed into $\mathbb{P}^2$.) We are interested in the size of $X(k)$. As a first approximation, it makes sense to compare it to the size of $\mathbb{P}^1(k)$, which equals $q + 1$.

Theorem 13.1.1 ("Riemann hypothesis", Hasse, Schmidt, Weil). *Suppose that X is a smooth and projective connected curve defined over $\mathbb{F}_q$, and let g be its genus. We have*

$$|\#X(k) - q - 1| \leq 2g\sqrt{q}.$$

It is called the Riemann hypothesis for the following reason. For any variety X over $k = \mathbb{F}_q$ one can organize the numbers $\#X(\mathbb{F}_{q^n})$ for varying $n \geq 1$ into the **Hasse–Weil zeta function**

$$\zeta_X(s) = \exp\left(\sum_{n \geq 1} \frac{\#X(\mathbb{F}_{q^n})}{n} q^{-ns} \right).$$

One can then show (by a theorem of Dwork, later reproved by Grothendieck) that $\zeta_X(s)$ is always a rational function of q^{-s}. Theorem 13.1.1 can then be seen to be equivalent to the statement that for X a smooth projective curve all zeros of $\zeta_X(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. Deligne has later generalized Theorem 13.1.1 to higher dimensional varieties.

Exercise 13.1.2. Compute $\zeta_{\mathbb{P}^n}(s)$.

We want to develop enough theory this term to prove Theorem 13.1.1. Let us give a quick preview how this will go. Notice first that $k = \mathbb{A}^1(k)$ is the solution set of the equation $x^q = x$ in $\bar{k}$, or equivalently the fixed-point set of the map $F : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $F(x) = x^q$. Note that $x^q - x = \prod_{\lambda \in \mathbb{F}_q}(x - \lambda)$, so there are no multiple roots, i.e. the fixed points all have multiplicity one. We can generalize this to our curve $X \subseteq \mathbb{P}^N$ by noting that the map

$$F : \mathbb{P}^N \to \mathbb{P}^N, \qquad F(x_0 : \cdots : x_N) = (x_0^q : \cdots : x_N^q)$$

maps X to X. Denoting the resulting map also by $F : X \to X$, we have

$$X(\mathbb{F}_q) = \mathrm{Fix}(F), \qquad X(\mathbb{F}_{q^n}) = \mathrm{Fix}(F^n).$$

The cardinalities of these sets can be realized geometrically as intersection numbers on the surface $X \times X$: we have

$$\#\mathrm{Fix}(F^n) = (\Delta_X . \Gamma_{F^n}),$$

the intersection multiplicity of two curves: the diagonal $\Delta_X \subseteq X \times X$ and the graph Γ_{F^n} of F^n. Algebraic geometry allows you to control these numbers by means of intersection theory, which is based on properties of invertible sheaves on curves.

13.2. Invertible sheaves on $\mathbb{A}^1$ and $\mathbb{P}^1$

By a locally free sheaf we shall mean a locally free *coherent* sheaf, i.e. an $\mathcal{O}_X$-module which is locally isomorphic to $\mathcal{O}_X^n$ for some integer $n \geq 0$. As before, we shall use the terms "locally free (coherent) sheaf" and "vector bundle" interchangeably. Similarly, an invertible sheaf will often be called a "line bundle."

Lemma 13.2.1 (Vector bundles on $\mathbb{A}^1$). *Every locally free sheaf on $\mathbb{A}^1$ is free (isomorphic to $\mathcal{O}_{\mathbb{A}^1}^r$ for some $r \geq 0$). In particular, $\mathrm{Pic}(\mathbb{A}^1) = 0$.*

Proof. Let $\mathcal{E}$ be a locally free sheaf on $\mathbb{A}^1$. Since $\mathbb{A}^1 = \mathrm{Spec}(k[T])$ is affine, we have $\mathcal{E} \simeq \tilde{M}$ for some finitely generated $k[T]$-module M. By the structure theorem for f.g. modules over PIDs we can write

$$M \simeq k[T]^{\oplus r} \oplus \bigoplus_{i=1}^{m} k[T]/(f_i), \qquad f_1, \ldots, f_r \in k[T], \qquad \deg(f_i) > 0.$$

We claim that $m = 0$ (so that M, and hence also $\mathcal{E}$, is free). Otherwise let $x \in k$ be a zero of f_1 and let $\mathfrak{m} = (T - x)$ be the corresponding maximal ideal. Then $\mathcal{E}_x \simeq M_{\mathfrak{m}}$, which is not a free module over $\mathcal{O}_{\mathbb{A}^1, x} \simeq k[T]_{\mathfrak{m}}$ since it has torsion. $\square$

Construction. Let now $X = \mathbb{P}^1$ which we write as

$$X = U_0 \cup U_1, \qquad U_0 = D(x_0) = \mathrm{Spec}(k[y_1]), \qquad U_1 = D(x_1) = \mathrm{Spec}(k[y_0]).$$

We have

$$U_{01} := U_0 \cap U_1 = \mathrm{Spec}(k[y_0, y_1]/(y_0 y_1 - 1)),$$

which is a Laurent polynomial ring in either y_0 or y_1. In particular,

$$\mathcal{O}^\times(U_{01}) = k^\times \times y_0^{\mathbb{Z}} = k^\times \times y_1^{\mathbb{Z}}.$$

For $g_{01} = \lambda y_0^d \in \mathcal{O}^\times(U_{01})$ we define an invertible sheaf $\mathcal{L}(g_{01})$ on $\mathbb{P}^1$ by gluing together the trivial invertible sheaves $\mathcal{O}_{U_i}$ on U_i, $i = 0, 1$ along the isomorphism

$$\mathcal{O}_{U_0}|_{U_{01}} = \mathcal{O}_{U_{01}} \xrightarrow{g_{01}} \mathcal{O}_{U_{01}} = \mathcal{O}_{U_1}|_{U_{01}}.$$

Lemma 13.2.2 (Line bundles on $\mathbb{P}^1$)**.**

(a) *Every invertible sheaf on $\mathbb{P}^1$ is of the form $\mathcal{L}(g_{01})$ for some $g_{01} \in \mathcal{O}^\times(U_{01})$.*

(b) *We have $\mathcal{L}(g_{01}) \otimes \mathcal{L}(g'_{01}) \simeq \mathcal{L}(g_{01}g'_{01})$.*

(c) *We have $\mathcal{L}(g_{01}) \simeq \mathcal{O}_{\mathbb{P}^1}$ if and only if $g_{01} \in k^\times$.*

Definition 13.2.3. For $d \in \mathbb{Z}$ we define $\mathcal{O}_{\mathbb{P}^1}(d) = \mathcal{L}(y_0^d)$.

Corollary 13.2.4. *We have $\operatorname{Pic}(\mathbb{P}^1) \simeq \mathbb{Z}$, generated by the class of $\mathcal{O}_{\mathbb{P}^1}(1)$.*

Proof of Lemma 13.2.2. (a) Follows from Lemma 13.2.1. (b) Obvious once you unpack it and recall that the canonical isomorphism $\mathcal{O} \otimes_{\mathcal{O}} \mathcal{O} \to \mathcal{O}$ is given by $f \otimes g \mapsto fg$. To prove (c), let us compute

$$\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{L}(g_{01})) = \Gamma(\mathbb{P}^1, \mathcal{L}(g_{01})).$$

A section $f \in \Gamma(\mathbb{P}^1, \mathcal{L}(g_{01})$ corresponds to a pair of sections

$$f_i \in \Gamma(U_i, \mathcal{L}(g_{01})|_{U_i}) = \Gamma(U_i, \mathcal{O}_{U_i}) = k[y_{1-i}], \qquad i = 0, 1$$

which agree on the overlap U_{01}, which translates into the condition

$$f_1 = g_{01}f_0.$$

Such a section f corresponds to an isomorphism $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{L}(g_{01})$ if and only if $f_i \in \mathcal{O}^\times(U_i)$ for $i = 0, 1$.

We can now show ($\Leftarrow$): if $g_{01} = \lambda \in k^\times$ is a nonzero constant, we can simply take $(f_0, f_1) = (1, \lambda)$. To show ($\Rightarrow$), we can now assume $g_{01} = y_0^d$ (this reduction is inessential for the calculation). We then have

$$\Gamma(\mathbb{P}^1, \mathcal{O}(d)) = \{f_0 \in k[y_1] : y_0^d f_0(y_0^{-1}) \in k[y_0]\}.$$

But for a polynomial $f \in k[y]$, we have that $y^d f(y^{-1})$ is also a polynomial if and only if $\deg(f) \le d$. So

$$\Gamma(\mathbb{P}^1, \mathcal{O}(d)) = k[y_1]_{\le d}$$

is the space of polynomials of degree $\le d$, which can be identified more intrinsically/symmetrically with the space $k[x_0, x_1]_d$ of homogeneous polynomials of degree d. In particular,

$$\dim \Gamma(\mathbb{P}^1, \mathcal{O}(d)) = \dim k[x_0, x_1]_d = \max\{0, d + 1\}.$$

Since $\Gamma(\mathbb{P}^1, \mathcal{O})$ has dimension one, we conclude that $\mathcal{O}(d)$ is not isomorphic to $\mathcal{O}$ for $d \ne 0$, showing ($\Leftarrow$). $\qquad \square$

Remark 13.2.5 (Vector bundles on $\mathbb{P}^1$)**.** Given that Lemma 13.2.1 describes not only invertible sheaves but all locally free sheaves, one is led to ask what are the locally free sheaves on $\mathbb{P}^1$. The answer is given by the following theorem of Birkhoff and independently Grothendieck: every locally free sheaf on $\mathbb{P}^1$ is isomorphic to the direct sum of $\mathcal{O}(d_i)$ for some integers $d_1, \ldots, d_r \in \mathbb{Z}$. If we follow our strategy of proof of Lemma 13.2.2, we will see that this theorem is equivalent to the following elementary (but still non-obvious) statement: *given a matrix $G_{01} \in \operatorname{GL}_r(k[y, y^{-1}])$, there exist integers $d_1, \ldots, d_r$ and two matrices $F_0 \in \operatorname{GL}_r(k[y])$, $F_1 \in \operatorname{GL}_r(k[y^{-1}])$ such that*

$$F_0 G_{01} F_1 = \operatorname{diag}(y^{d_1}, \ldots, y^{d_r}).$$

13.3. Invertible sheaves on $\mathbb{A}^n$ and $\mathbb{P}^n$

We now redo everything we did in the previous subsection for $\mathbb{A}^n$ and $\mathbb{P}^n$. This is a bit more tricky but the results are essentially the same.

Lemma 13.3.1 (Line bundles on $\mathbb{A}^n$)**.** *We have* $\operatorname{Pic}(\mathbb{A}^n) = 0$.

Proof (sketch). The proof relies on the following (not so difficult) fact from commutative algebra: *Every invertible (nonzero and locally principal) in a unique factorization domain is principal.* We can apply it since $A := \mathcal{O}(\mathbb{A}^n) = k[x_1,\ldots,x_n]$ is a UFD. Let $\mathcal{L}$ be an invertible sheaf on $\mathbb{A}^n$ and write it as $\tilde{L}$ for an A-module L. Let $x \in L$ be a nonzero element, corresponding to a map $A \to L$. We dualize it (apply $\operatorname{Hom}(-,A)$) to get a map $L^\vee \to A$ which is injective (as this can be checked locally, whereupon $L = A$ and the map is multiplication by a nonzero $x \in A$). The image $I \subseteq A$ of this map is then a nonzero locally principal ideal (again, check locally), and hence principal. Thus $L^\vee \simeq I$ is free and hence so is L. $\qquad\square$

Remark 13.3.2 (Vector bundles on $\mathbb{A}^n$)**.** Given that Lemma 13.2.1 describes not only invertible sheaves but all locally free sheaves, one is led to ask what are the locally free sheaves on $\mathbb{A}^n$. The answer is given by the following theorem of Quillen and Suslin (answering a long-standing conjecture of Serre): every locally free sheaf on $\mathbb{A}^n$ is free. Equivalently, every finitely generated module over the polynomial ring is free.

Construction. Let now $X = \mathbb{P}^n$, which we write as

$$X = \bigcup_{i=0}^{n} U_i, \qquad U_i = D(x_i) = \operatorname{Spec}(k[y_{0/i},\ldots,y_{n/i}]/(y_{i/i} - 1))$$

using Vakil's handy notation $y_{j/i}$ for the regular function x_j/x_i on U_i. For a non-empty subset $I \subseteq \{0,\ldots,n\}$ we write

$$U_I = \bigcap_{i \in I} U_i = \operatorname{Spec}\left(\frac{k[y_{j/i} \,(\text{for } j \in \{0,\ldots,n\}, i \in I)]}{(y_{i/i} - 1 \,(\text{for } i \in I),\ y_{j/i'}y_{i'/i} - y_{j/i} \,(\text{for } j \in \{0,\ldots,n\}, i, i' \in I))} \right).$$

This is isomorphic (after permuting the variables so that $I = \{0,\ldots,\#I - 1\}$) to $\mathbb{G}_m^{\#I-1} \times \mathbb{A}^{n+1-\#I}$. We have

$$\mathcal{O}^\times(U_I) = k^\times \times \prod_{i < i'} y_{i/i'}^{\mathbb{Z}}.$$

In order to describe an invertible sheaf $\mathcal{L}$ on $\mathbb{P}^n$, it suffices to describe its restrictions $\mathcal{L}_i = \mathcal{L}|_{U_i}$ and isomorphisms $\phi_{ij}: \mathcal{L}_i|_{U_{ij}} \to \mathcal{L}_j|_{U_{ij}}$ (with $\phi_{ii} = \operatorname{id}$) which are "transitive" i.e. which satisfy the cocycle condition

$$\phi_{ij} \circ \phi_{jk} \circ \phi{ki} = \operatorname{id} \quad \text{on } \mathcal{L}|_{U_{ijk}}$$

(this condition did not appear for $\mathbb{P}^1$ since we had only two open sets and no interesting triple overlaps). Now thanks to 13.3.1 we have $\mathcal{L}_i \simeq \mathcal{O}_{U_i}$ (this isomorphism is canonical up to $\operatorname{Aut}(\mathcal{O}_{U_i}) = \mathcal{O}^\times(U_i) = k^\times$). After making such an identification, the maps ϕ_{ij} are given by multiplication by some $g_{ij} \in \mathcal{O}^\times(U_{ij}) = k^\times \times y_{i/j}^{\mathbb{Z}}$, satisfying $g_{ii} = 1$ and $g_{ij}g_{jk}g_{ki} = 1$ on U_{ijk}. If we change each identification $\mathcal{L}_i \simeq \mathcal{O}_{U_i}$ by an $f_i \in \mathcal{O}^\times(U_i)$, the g_{ij} are replaced with $g_{ij} \cdot (f_i/f_j)$.

Remark 13.3.3. The above considerations show that for every scheme or algebraic set X and an open cover $X = \bigcup U_i$ such that $\operatorname{Pic}(U_i) = 0$, the Picard group $\operatorname{Pic}(X)$ its the cohomology of the two-term

sequence

$$(g_{ij}) \qquad \mapsto \qquad (g_{ij}g_{jk}g_{ki})$$
$$\cap \qquad\qquad\qquad \cap$$
$$\textstyle\prod_i \mathcal{O}^\times(U_i) \longrightarrow \prod_{ij} \mathcal{O}^\times(U_{ij}) \longrightarrow \prod_{ijk} \mathcal{O}^\times(U_{ijk})$$
$$\cup \qquad\qquad\qquad \cup$$
$$(f_i) \qquad \mapsto \qquad (f_i/f_j)$$

This sequence is part of the Čech complex for the sheaf $\mathcal{O}^\times$ and the open cover $\{U_i\}$. Once we discuss cohomology, we shall relate this to the identification

$$\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}^\times)$$

with the first cohomology group of $\mathcal{O}^\times$, valid with no assumptions on $\mathrm{Pic}(U_i)$.

We now finish our construction.

Definition 13.3.4. Let $d \in \mathbb{Z}$. We define the invertible sheaf $\mathcal{O}(d)$ on $\mathbb{P}^n$ using the functions $(g_{ij}) \in \mathcal{O}^\times(U_{ij})$ given by

$$g_{ij} = (x_i/x_j)^d = y_{i/j}^d = y_{j/i}^{-d}.$$

It is clear that the cocycle condition is satisfied.

Lemma 13.3.5. *Let $n \geq 1$.*

(a) *Every invertible sheaf on $\mathbb{P}^n$ is of the form $\mathcal{O}_{\mathbb{P}^n}(d)$ for a unique $d \in \mathbb{Z}$.*

(b) *We have $\mathcal{O}(d) \otimes \mathcal{O}(d') \simeq \mathcal{O}(d + d')$.*

(c) *We have*

$$\Gamma(\mathbb{P}^n, \mathcal{O}(d)) \simeq k[x_0, \ldots, x_n]_d$$

which is a k-vector space of dimension $\binom{n+d}{d}$.

Proof. (b) is clear, and (c) is a direct calculation which we omit. From (c) it follows that $\dim \Gamma(\mathbb{P}^n, \mathcal{O}(d))$ equals 1 only for $d \neq 0$, showing uniqueness in (a). The remainder of (a) will be shown later, once we discuss the relationship between invertible sheaves and divisors. $\square$

Remark 13.3.6 (Vector bundles on $\mathbb{P}^n$)**.** Given that Remark 13.2.5 describes all locally free sheaves on $\mathbb{P}^1$ and Lemma 13.3.5 says that all invertible sheaves are of the form $\mathcal{O}(d)$, one may wonder what one can say about higher rank locally free sheaves on $\mathbb{P}^n$ for $n > 1$. The answer, in brief, is that vector bundles on $\mathbb{P}^n$ of rank r are **hard** to describe as soon as $n > 1$ or $r > 1$. In particular, not every locally free sheaf on $\mathbb{P}^n$ is **split**, i.e. of the form $\mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_r)$ for some integers $d_1, \ldots, d_r$. For example, the cotangent bundle (sheaf of Kähler differentials) $\Omega^1_{\mathbb{P}^n}$ is of rank n and is not split if $n > 1$ (this is very easy to prove using coherent cohomology, a topic we shall treat next). In fact, already rank two vector bundles on $\mathbb{P}^2$ can vary in families, by which we mean: there exists a positive-dimensional variety S and a rank two locally free sheaf $\mathcal{E}$ on $\mathbb{P}^2 \times S$ such that for every two distinct points $s_0, s_1 \in S$ the restrictions of $\mathcal{E}$ to $\{s_i\} \times \mathbb{P}^2$ ($i = 0, 1$) are not isomorphic as locally free sheaves on $\mathbb{P}^2$. In contrast, an important and widely open conjecture of Hartshorne states that rank two vector bundles on $\mathbb{P}^n$ are split as long as $n \geq 5$. It is rather embarrassing that we lack an answer to such a simple looking question.