12. Lecture 12 (Mar 10): Ideal sheaves and locally free sheaves

Recommended reading: Kempf §5, Hartshorne II.5

12.1. Closed immersions and sheaves of ideals

A **quasi-coherent ideal** is a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{I}$ of $\mathcal{O}_X$. The quotient $\mathcal{O}_X/\mathcal{I}$ is then a quasi-coherent $\mathcal{O}_X$-algebra. If X is a locally noetherian scheme or an algebraic set, then every quasi-coherent ideal of $\mathcal{O}_X$ is coherent.

Lemma 12.1.1. *Let $i\colon Y \to X$ be a closed immersion. Then the kernel $\mathcal{I}_Y$ of the surjection $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is a quasi-coherent ideal. Moreover, i is a finite morphism. Conversely, for every quasi-coherent ideal $\mathcal{I}$ there exists a unique closed immersion $i\colon Y \to X$ for which $i_*\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}$.*

Proof. The sheaf $i_*\mathcal{O}_Y$ is quasi-coherent by Lemma 11.1.3, and hence $\mathcal{I}$ is quasi-coherent, being the kernel of a map between quasi-coherent sheaves. Knowing $\mathcal{I}$, we can reconstruct Y as the support of $\mathcal{O}_X/\mathcal{I}$ (a closed subset of X) together with the restriction $\mathcal{O}_Y$ of the sheaf $\mathcal{O}_X/\mathcal{I}$. On an affine open $U = \operatorname{Spec}(A) \subseteq X$, we have $i^{-1}(U) \simeq \operatorname{Spec}(A/I)$ where $I = \mathcal{I}(U)$. In particular, i is an affine (and hence finite) morphism. □

Corollary 12.1.2. *If $X = \operatorname{Spec}(A)$, then every closed subscheme of X is of the form $\operatorname{Spec}(A/I)$ for a unique ideal $I \subseteq A$.*

12.2. Locally free sheaves (a.k.a. vector bundles)

A locally free $\mathcal{O}_X$-module is an algebraic analog of a vector bundle over X.

Definition 12.2.1. Let X be a ringed space. An $\mathcal{O}$-module $\mathcal{E}$ is **locally free**[6] if it is locally isomorphic to $\mathcal{O}_X^n$ for some integer $n \geq 0$. More precisely, if there exists an open cover $X = \bigcup U_\alpha$, integers $n_\alpha \geq 0$, and isomorphisms $\mathcal{E}|_{U_\alpha} \simeq \mathcal{O}_{U_\alpha}^n$. If all n_α are equal to a single integer n, we say that $\mathcal{E}$ has **rank** n.

Lemma 12.2.2. *Let X be a locally noetherian scheme or an algebraic set and let $\mathcal{E}$ be a coherent sheaf on X. The following conditions are equivalent:*

- *(a) $\mathcal{E}$ is locally free;*
- *(b) for every $x \in X$, the stalk $\mathcal{E}_x$ is a free $\mathcal{O}_{X,x}$-module;*
- *(c) for every affine open $U \subseteq X$, $\mathcal{E}(U)$ is a projective $\mathcal{O}(U)$-module.*

If X is reduced (or an algebraic set), these are further equivalent to

- *(d) The function*
$$\rho(x) = \dim_{\kappa(x)} \mathcal{E}(x)$$
is locally constant.

Example 12.2.3. The most natural example of a locally free sheaf is the sheaf of algebraic differential forms Ω_X^1 on a smooth variety X (see Lecture 17).

Remark 12.2.4. Tensor operations from (multi)linear algebra (duals, tensor product, symmetric and exterior powers) all apply to locally free sheaves without change.

Remark 12.2.5. The algebra of polynomial functions on a vector space V is the symmetric algebra $\operatorname{Sym}(V^\vee)$. We can therefore turn V into a scheme $\operatorname{Spec}(\operatorname{Sym}(V^\vee))$. Using this recipe, we can turn a locally free sheaf $\mathcal{E}$ into an honest vector bundle $E \to X$, a map whose fiber at $x \in X$ is naturally isomorphic to the $\kappa(x)$-vector space $E(x)$, as follows:
$$E = \mathbf{Spec}_X(\operatorname{Sym}(\mathcal{E}^\vee)).$$

[6] This terminology may be non-standard, as we require our locally free sheaves to have locally finite rank.

12.3. Invertible sheaves (a.k.a. line bundles)

Invertible sheaves are locally free sheaves of rank one, and therefore correspond (and are often called) line bundles.

Definition 12.3.1. An **invertible sheaf** is a locally free sheaf of rank one, i.e. an $\mathcal{O}_X$-module $\mathcal{L}$ locally isomorphic to $\mathcal{O}_X$.

The terminology "invertible" comes from the fact that invertible sheaves are invertible elements of the monoid of (isomorphism classes of) $\mathcal{O}_X$-modules under the tensor product. Namely, if $\mathcal{L}$ is an invertible sheaf, then so is $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$, and the "evaluation map" $\mathcal{L} \otimes \mathcal{L}^\vee \to \mathcal{O}_X$ is an isomorphism.

If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves, then so is their tensor product $\mathcal{L} \otimes \mathcal{M}$. Thus, the isomorphism classes of invertible sheaves on X form an abelian group (with neutral element $\mathcal{O}_X$ and tensor product as the group operation) called the **Picard group** of X and denoted by $\mathrm{Pic}(X)$. Thinking in terms of this group, it is reasonable to introduce the following notation: for an invertible sheaf $\mathcal{L}$ and an integer $n \in \mathbb{Z}$, we set $\mathcal{L}^n = \mathcal{L} \otimes \cdots \otimes \mathcal{L}$ (n factors) if $n \geq 1$, $\mathcal{L}^0 = \mathcal{O}_X$, and $\mathcal{L}^n = (\mathcal{L}^\vee)^{-n}$ if $n < 0$. We shall sometimes use the notation $\mathcal{L}^{\otimes n}$ for $\mathcal{L}^n$, to avoid confusing it with the direct sum $\mathcal{L}^{\oplus n} = \mathcal{L} \oplus \cdots \oplus \mathcal{L}$.

Remark 12.3.2 (Invertible sheaves via cocycles)**.** Invertible sheaves are conveniently described in terms of Čech cocycles valued in the sheaf $\mathcal{O}_X^\times$ of nowhere vanishing functions. Namely, if $\mathcal{L}$ is an invertible sheaf, we can find an open cover $X = \bigcup U_\alpha$ and isomorphisms $\phi_\alpha \colon \mathcal{O}_{U_\alpha} \xrightarrow{\sim} \mathcal{L}|_{U_\alpha}$. Then on $U_\alpha \cap U_\beta$, the composition $\phi_\beta^{-1}\phi_\alpha$ gives an $\mathcal{O}$-module isomorphism $\mathcal{O}_{U_\alpha \cap U_\beta} \to \mathcal{O}_{U_\alpha \cap U_\beta}$, which is the multiplication by a nonvanishing function $g_{\alpha\beta} \in \mathcal{O}^\times(U_\alpha \cap U_\beta)$. The tuple of functions $\{g_{\alpha\beta}\}$ satisfies the cocycle condition

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$. Such a tuple is called a Čech cocycle valued in the sheaf $\mathcal{O}^\times$. Conversely, given an open cover $X = \bigcup U_\alpha$ and a Čech cocycle $\{g_{\alpha\beta}\}$, we can construct an invertible sheaf $\mathcal{L}$ by gluing together the sheaves $\mathcal{O}_{U_\alpha}$ along the isomorphism $g_{\alpha\beta}$ on the overlaps $U_\alpha \cap U_\beta$. If $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves, given by cocycles $\{g_{\alpha\beta}\}$ and $\{h_\alpha \beta\}$ subordinate to the same open cover $X = \bigcup U_\alpha$, then the invertible sheaves $\mathcal{L}^n$ and $\mathcal{L} \otimes \mathcal{M}$ are given by the cocycles

$$\{g_{\alpha\beta}^n\} \quad \text{and} \quad \{g_{\alpha\beta} h_{\alpha\beta}\},$$

respectively. (Once we learn about cohomology, a more detailed analysis of this will show that $\mathrm{Pic}(X)$ is naturally isomorphic to the cohomology group $H^1(X, \mathcal{O}_X^\times)$.)

Example 12.3.3 (The sheaves $\mathcal{O}(d)$ on $\mathbb{P}^n$)**.** Let $\pi \colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the projection map and let d be an integer. For an open subset $V \subseteq \mathbb{A}^{n+1}$ which is invariant under the action of $k^\times$ by scaling, we say that a function $f \in \mathcal{O}(V)$ is homogeneous of degree d if $f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$ holds for all $\lambda \in k^\times$ and all $(x_0, \ldots, x_n) \in V$. For an open subset $U \subseteq \mathbb{P}^n$, we set

$$\mathcal{O}(d)(U) = \{f \in \mathcal{O}(\pi^{-1}(V)) : f \text{ is homogeneous of degree } d\}.$$

This is a sheaf, by construction a subsheaf of $\pi_*\mathcal{O}$.

Let us compute $\mathcal{O}(d)(D_\mathbb{P}(f))$ for a distinguished affine open $D_\mathbb{P}(f) \subseteq \mathbb{P}^n$, with $f \in k[x_0, \ldots, x_n]$ homogeneous of degree $e > 0$. We have $\pi^{-1}(D_\mathbb{P}(f)) = D(f)$, the distinguished affine open in $\mathbb{A}^{n+1}$, and hence

$$\mathcal{O}(\pi^{-1}(D_\mathbb{P}(f))) = k[x_0, \ldots, x_n][f^{-1}].$$

This is naturally a graded ring, with homogeneous elements of degree d of the form h/f^m where $h \in k[x_0, \ldots, x_n]$ is homogeneous of degree $d + me$.

In the special case $f = x_0$, we have

$$\mathcal{O}(d)(D(x_0)) = k[x_0, \ldots, x_n][x_0^{-1}]_d = x_0^d k[x_1/x_0, \ldots, x_n/x_0],$$

a free $\mathcal{O}(D(x_0))$-module of rank one.

We deduce from these computations that $\pi_*\mathcal{O} = \bigoplus_{d\in\mathbb{Z}} \mathcal{O}(d)$ and that each $\mathcal{O}(d)$ is an invertible sheaf. We shall study them in greater detail in Lecture 13.