11. Lecture 11 (Feb 26): Quasi-coherent sheaves

Recommended reading: Kempf §5, Hartshorne II.5

11.1. More on quasi-coherent sheaves

Lemma 11.1.1. *Let $X = \mathrm{Spec}(A)$ be an affine scheme and let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ is quasi-coherent, then

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0$$

is exact.

Proof. As sequence is always left-exact, we have to show that the right map is surjective. Let $s \in \mathcal{F}''(X)$. Since $\mathcal{F} \to \mathcal{F}''$ is surjective, there exists an open cover $X = \bigcup_{\alpha \in I} U_\alpha$ such that $s|_{U_\alpha}$ lies in the image of $\mathcal{F}(U_\alpha) \to \mathcal{F}''(U_\alpha)$ for every $\alpha \in I$. Since X is quasi-compact and distinguished affine opens form a base, we may assume that the index set I is finite and that $U_\alpha = D(f_\alpha)$ for some elements $f_\alpha \in A$ generating the unit ideal. Refining the cover further, we may assume that in addition the restriction of $\mathcal{F}'$ to each U_α is of the form $\tilde{M}_\alpha$ for some $\mathcal{O}(U_\alpha)$-module M_α. Now consider the commutative diagram of Čech complexes:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}'(X) & \longrightarrow & \prod_{\alpha\in I}\mathcal{F}'(U_\alpha) & \longrightarrow & \prod_{\alpha,\beta\in I}\mathcal{F}'(U_\alpha\cap U_\beta) & \longrightarrow & \prod_{\alpha,\beta,\gamma\in I}\mathcal{F}'(U_\alpha\cap U_\beta\cap U_\gamma) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \prod_{\alpha\in I}\mathcal{F}(U_\alpha) & \longrightarrow & \prod_{\alpha,\beta\in I}\mathcal{F}(U_\alpha\cap U_\beta) & \longrightarrow & \prod_{\alpha,\beta,\gamma\in I}\mathcal{F}(U_\alpha\cap U_\beta\cap U_\gamma) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}''(X) & \longrightarrow & \prod_{\alpha\in I}\mathcal{F}''(U_\alpha) & \longrightarrow & \prod_{\alpha,\beta\in I}\mathcal{F}''(U_\alpha\cap U_\beta) & \longrightarrow & \prod_{\alpha,\beta,\gamma\in I}\mathcal{F}''(U_\alpha\cap U_\beta\cap U_\gamma) & \longrightarrow & \dots
\end{array}$$

We make the following observations:

- The diagram commutes (since the formation of the Čech complex is functorial).
- The columns are exact (by left-exactness of taking sections), i.e. exact after forgetting the bottom zero.
- All three rows are exact until the third column (the one with $\prod_{\alpha,\beta\in I}$). This is the sheaf condition.
- By Lemma 10.2.4 we know that $\mathcal{F}'$ is of the form $\tilde{M}$ for some A-module M, and hence the top row (the one with $\mathcal{F}'$) is exact, by Lemma 9.1.1 proved earlier.

To finish the argument, we perform a tedious diagram chase which is best explained on the blackboard or done by oneself. We shall use the symbol d for the horizontal (Čech) differentials. By assumption, the image $(s|_{U_\alpha}) \in \prod_{\alpha\in I}\mathcal{F}''(U_\alpha)$ of $s \in \mathcal{F}''(X)$ is the image of an element $(t_\alpha) \in \prod_{\alpha\in I}\mathcal{F}(U_\alpha)$. If $d(t_\alpha) = 0 \in \prod_{\alpha,\beta\in I}\mathcal{F}(U_\alpha\cap U_\beta)$, then by exactness of the middle row it comes from a unique $t \in \mathcal{F}(X)$, which then maps to $s \in \mathcal{F}''(X)$ and we are done. In general, the image

$$d(t_\alpha) = (t_\alpha|_{U_\alpha\cap U_\beta} - t_\beta|_{U_\alpha\cap U_\beta}) \in \prod_{\alpha,\beta\in I}\mathcal{F}(U_\alpha\cap U_\beta)$$

maps via the vertical map to

$$d(s_\alpha) = (s|_{U_\alpha \cap U_\beta} - s|_{U_\alpha \cap U_\beta}) = 0 \in \prod_{\alpha,\beta \in I} \mathcal{F}''(U_\alpha \cap U_\beta),$$

and hence by exactness of the third column it is the image of an element $(u_{\alpha\beta}) \in \prod_{\alpha,\beta\in I} \mathcal{F}'(U_\alpha \cap U_\beta)$. Again, the image of $d(u_{\alpha\beta})$ in $\prod_{\alpha,\beta,\gamma\in I} \mathcal{F}(U_\alpha \cap U_\beta \cap U_\gamma)$ is $d(d(t_\alpha)) = 0$. By the injectivity of the vertical maps from the 1st to the 2nd row, we deduce that $d(u_{\alpha\beta}) = 0$. By exactness of the 1st row (here we use the additional assumption $\mathcal{F}' = \tilde{M}$) we obtain a $(u_\alpha) \in \prod_{\alpha\in I} \mathcal{F}'(U_\alpha)$ such that

$$(u_\alpha|_{U_\alpha \cap U_\beta} - u_\beta|_{U_\alpha \cap U_\beta}) = d(u_\alpha) = (u_{\alpha\beta}).$$

Now set

$$t'_\alpha = t_\alpha - u_\alpha.$$

As u_α maps to zero in $\mathcal{F}''(U_\alpha)$, the image of (t'_α) in $\prod_{\alpha\in I} \mathcal{F}''(U_\alpha)$ equals (s_α). We check that

$$d(t'_\alpha) = d(t_\alpha) - d(u_\alpha) = d(t_\alpha) - (u_{\alpha\beta}) = 0,$$

and again we deduce the existence of a unique $t' \in \mathcal{F}(X)$ whose image in $\mathcal{F}''(X)$ equals s. □

Corollary 11.1.2. *Let X be a scheme or an algebraic set and let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent, then so is $\mathcal{F}$.

Proof. The assertion is local so we may assume that X is affine. By Lemma 11.1.1, the sequence of $\mathcal{O}(X)$-modules

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0$$

is exact. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathcal{F}'(X)} & \longrightarrow & \widetilde{\mathcal{F}(X)} & \longrightarrow & \widetilde{\mathcal{F}''(X)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \end{array}$$

in which the vertical arrows are the counit maps (see Lemma 10.2.4). The top row is exact by exactness of the functor $\widetilde{(-)}$. Since $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent, the left and right maps are isomorphisms. The five lemma implies that so is the middle one, so that $\mathcal{F}$ is quasi-coherent. □

Functorial properties of quasi-coherent sheaves:

Lemma 11.1.3. *Let $f\colon Y \to X$ be a map of schemes or algebraic sets.*

(a) For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the pull-back $f^*\mathcal{F}$ is quasi-coherent.

(b) The same holds for coherent sheaves if X and Y are algebraic sets or locally noetherian schemes.

(c) Suppose that X and Y are noetherian schemes[5] or algebraic sets. Then for every quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$, the push-forward $f_*\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module.

[5]More generally the proof shows the claim if f is quasi-compact and quasi-separated [SP Section 01KH].

Proof. Assertions (a) and (b) are local on Y, and hence are easily reduced to the case X and Y affine, in which situation we might apply Lemma 10.1.6(a). For (c), this is local on X, and we may assume that X is affine. If Y is also affine, we win by Lemma 10.1.6(b). In general, since Y is quasi-compact, we can cover it by finitely many affines U_α ($\alpha = 1, \dots, n$). Since Y is a noetherian topological space, the intersections $U_\alpha \cap U_\beta$ are quasi-compact, so we can cover them by finitely many affines $U_{\alpha\beta\gamma}$ ($\gamma = 1, \dots, m(\alpha, \beta)$). Let $j_\alpha \colon U_\alpha \to X$ and $j_{\alpha\beta\gamma} \colon U_{\alpha\beta\gamma} \to X$ be the inclusions. We can now express $\mathcal{F}$ as the kernel

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_\alpha j_{\alpha,*} j_\alpha^* \mathcal{F} \longrightarrow \bigoplus_{\alpha,\beta,\gamma} j_{\alpha\beta\gamma,*} j_{\alpha\beta\gamma}^* \mathcal{F},$$

(technically these are more canonically the products not direct sums, but since the index sets are finite, it does not matter). Since f_* is left-exact and commutes with direct sums, we obtain a description of $f_*\mathcal{F}$ as the kernel

$$\begin{array}{ccccccc} 0 \longrightarrow & f_*\mathcal{F} & \longrightarrow & f_* \bigoplus_\alpha j_{\alpha,*} j_\alpha^* \mathcal{F} & \longrightarrow & f_* \bigoplus_{\alpha,\beta,\gamma} j_{\alpha\beta\gamma,*} j_{\alpha\beta\gamma}^* \mathcal{F} \\ & & & \| & & \| \\ & & & \bigoplus_\alpha (f \circ j_\alpha)_* (\mathcal{F}|_{U_\alpha}) & \longrightarrow & \bigoplus_{\alpha\beta\gamma} (f \circ j_{\alpha\beta\gamma})_* (\mathcal{F}|_{U_{\alpha\beta\gamma}}). \end{array}$$

Since each $f \circ j_\alpha$ and $f \circ j_{\alpha\beta\gamma}$ is a map between affines, every summand above is quasi-coherent, and hence so is $\mathcal{F}$. ☐

11.2. Fibers of coherent sheaves

Definition 11.2.1. Let X be a scheme or an algebraic set and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The **fiber** of $\mathcal{F}$ at a point $x \in X$ is

$$\mathcal{F}(x) \in \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x),$$

a vector space over the residue field $\kappa(x)$.

As a corollary of Nakayama's lemma, we have:

Lemma 11.2.2. *Let X be a locally noetherian scheme or an algebraic set and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then:*

(a) $\dim_{\kappa(x)} \mathcal{F}(x) < \infty$ *for all $x \in X$;*

(b) a map of quasi-coherent sheaves $\mathcal{F}' \to \mathcal{F}$ is surjective if and only if the maps $\mathcal{F}'(x) \to \mathcal{F}(x)$ are surjective for all $x \in X$;

(c) $\mathcal{F} = 0$ if and only if $\mathcal{F}(x) = 0$ for all $x \in X$.

Moreover, in the case of a scheme, it is enough to consider only closed points in (b) and (c).

Proof. Let $x \in X$ and let $M = \mathcal{F}_x$, which is a finitely generated module over the local ring $\mathcal{O}_{X,x}$, and then $\mathcal{F}(x) = M \otimes \kappa(x)$ is of finite dimension, showing (a). By Nakayama's lemma, a map of $\mathcal{O}_{X,x}$-modules $N \to M$ is surjective if and only if $N \otimes \kappa(x) \to M \otimes \kappa(x)$ is surjective. Since a map of sheaves is surjective if and only if it is surjective on stalks, this shows (b), and (c) follows as a special case $\mathcal{F}' = 0$. As for the "moreover" part, we may assume that X is affine and that $\mathcal{F} = \tilde{M}$ for a finitely generated module M over $A = \mathcal{O}(X)$. The set $\mathrm{supp}(M)$ of all primes $\mathfrak{p} \subseteq A$ such that $M_\mathfrak{p} \neq 0$ is a closed subset of $\mathrm{Spec}(A)$ (as it is equal to $V(\mathrm{Ann}(A))$ where $\mathrm{Ann}(A) = \{a \in A : aM = 0\}$). This shows the stronger form of (c), and the stronger form of (a) follows similarly if we consider the cokernel of $\mathcal{F}' \to \mathcal{F}$ in place of $\mathcal{F}$. ☐

11.3. Affine and finite morphisms

Recall from Lecture 9 that a morphism of schemes or algebraic sets $\phi\colon Y \to X$ is **affine** if for every affine $U \subseteq X$, the preimage $\phi^{-1}(U) \subseteq Y$ is affine. It is **finite** if moreover $\mathcal{O}(\phi^{-1}(U))$ is a finite $\mathcal{O}(U)$-module for every affine $U \subseteq X$.

Proposition 11.3.1. *(a) Let X be a scheme. For every affine morphism $\phi\colon Y \to X$, the push-forward $\phi_*\mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_X$-algebra. The construction $(\phi\colon Y \to X) \mapsto \phi_*\mathcal{O}_Y$ defines an equivalence of categories*

$$\{\textit{affine morphisms } Y \to X\}^{\mathrm{op}} \xrightarrow{\sim} \{\textit{quasi-coherent } \mathcal{O}_X\textit{-algebras}\}.$$

(b) If X is locally noetherian, the same construction gives an equivalence

$$\{\textit{finite morphisms } Y \to X\}^{\mathrm{op}} \xrightarrow{\sim} \{\textit{coherent } \mathcal{O}_X\textit{-algebras}\}.$$

(c) A morphism $\phi\colon Y \to X$ is affine (resp. finite) if and only if there exists an affine open cover $X = \bigcup U_\alpha$ such that the preimages $\phi^{-1}(U_\alpha)$ are affine (resp. and $cO(\phi^{-1}(U_\alpha))$ is finite over $\mathcal{O}(U_\alpha)$) for every α.

(d) In particular, if X is affine, then $Y \to X$ is affine (resp. finite) if and only if Y is affine (resp. and $\mathcal{O}(Y)$ is finite over $\mathcal{O}(X)$).

The inverse functor is called the **relative spectrum** $\mathcal{A} \mapsto \mathbf{Spec}_X(\mathcal{A})$.

Remark 11.3.2. Let X be an algebraic set. Then similarly we have equivalences as above,

$$\{\text{affine morphisms } Y \to X\}^{\mathrm{op}} \xrightarrow{\sim} \{\text{reduced quasi-coherent } \mathcal{O}_X\text{-algebras of finite type}\}.$$

and

$$\{\text{finite morphisms } Y \to X\}^{\mathrm{op}} \xrightarrow{\sim} \{\text{reduced coherent } \mathcal{O}_X\text{-algebras}\}.$$

Example 11.3.3 (Normalization)**.** Let X be an integral scheme or an irreducible algebraic set, let $K = k(X)$ be its function field, and let L/K be a finite extension of fields. Consider the $\mathcal{O}_X$-algebra $\mathcal{A}$ defined on non-empty affine opens $U \subseteq X$ by

$$\mathcal{A}(U) = \text{integral closure of } \mathcal{O}(U) \text{ in } L.$$

Then $\mathcal{A}$ is a quasi-coherent algebra, and the corresponding affine morphism $Y \to X$ is called the **normalization of X in L**. By "finiteness of integral closure," if X is an irreducible algebraic set, a scheme of finite type over a field, or a locally noetherian scheme and L/K is separable, then the morphism $Y \to X$ is finite. We shall primarily use this construction in two cases: $L = K$ (in which case we call Y the normalization of X), or when X is a smooth curve.

Example 11.3.4. Let $\pi\colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the projection map. The preimage of the standard affine open $U_i = D(x_i) \simeq \mathbb{A}^n$ is the open subset $D(x_i) \subseteq \mathbb{A}^{n+1}$. Thus π is affine. The corresponding coherent $\mathcal{O}_{\mathbb{P}^n}$-algebra is the direct sum $\bigoplus_{d\in\mathbb{Z}} \mathcal{O}(d)$ (see the next lecture).