10. Lecture 10 (Feb 24): Sheaves of modules

Our treatment of quasi-coherent and coherent sheaves largely follows Hartshorne II.5, except that we replace some of the hands-on computations in the proof with the slightly more conceptual arguments using the exactness of the Čech complex proved in Lecture 9. (Our definition of the sheaves $\mathcal{O}(d)$ in $\mathbb{P}^n$ is also different, since we avoid discussing the link between coherent sheaves on $\mathbb{P}^n$ and graded modules over the polynomial ring.)

Recommended reading: Kempf §5, Hartshorne II.5

10.1. Sheaves of modules

Definition 10.1.1. Let $\mathcal{O}$ be a sheaf of (commutative, unital) rings on a topological space X. An **$\mathcal{O}$-module** is a sheaf of abelian groups $\mathcal{M}$ together with a map of sheaves

$$\mathcal{O} \times \mathcal{M} \longrightarrow \mathcal{M}$$

giving $\mathcal{M}(U)$ the structure of an $\mathcal{O}(U)$-module for every open $U \subseteq X$. A morphism of $\mathcal{O}$-modules is a map of sheaves $f\colon \mathcal{F} \to \mathcal{G}$ such that $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}(U)$-modules for every open $U \subseteq X$.

Remarks 10.1.2. (a) Equivalent characterizations of the structure of an $\mathcal{O}$-module on a sheaf of abelian groups $\mathcal{M}$:

- for every $U \subseteq X$, the structure of an $\mathcal{O}(U)$-module on $\mathcal{M}(U)$ such that the restriction maps $\mathcal{M}(U) \to \mathcal{M}(V)$ are maps of $\mathcal{O}(U)$-modules; or
- a homomorphism of sheaves of rings $\mathcal{O} \longrightarrow \mathcal{H}om(\mathcal{M},\mathcal{M})$.

(b) Many sources call an $\mathcal{O}$-module a "sheaf of $\mathcal{O}$-modules," though that is a bit redundant as there is no other possible meaning of "$\mathcal{O}$-module."

(c) Familiar notions from commutative algebra have natural $\mathcal{O}$-module analogs. Kernels, cokernels, images work in the obvious way. Tensor product is identified as the sheafification of the obvious tensor product of presheaves $U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{M}'(U)$. Further examples: an ideal is an $\mathcal{O}$-submodule $\mathcal{I} \subseteq \mathcal{O}$, and an $\mathcal{O}$-algebra is a map $\mathcal{O} \to \mathcal{A}$ of sheaves of rings, or equivalently an $\mathcal{O}$-module $\mathcal{A}$ equipped with a map $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A} \to \mathcal{A}$ satisfying suitable axioms. For a map of sheaves of rings $\mathcal{O} \to \mathcal{O}'$, there is the usual adjunction between the forgetful functor $\mathbf{Mod}_{\mathcal{O}'} \to \mathbf{Mod}_{\mathcal{O}}$ (right adjoint) and the tensor product $(-) \otimes_{\mathcal{O}} \mathcal{O}'\colon \mathbf{Mod}_{\mathcal{O}} \to \mathbf{Mod}_{\mathcal{O}'}$ (left adjoint).

(d) If it is understood from the context that we work with $\mathcal{O}_X$-modules on a ringed space X, we write $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ (resp. $\mathcal{H}om(\mathcal{F},\mathcal{G})$) for the set (resp. sheaf) of all $\mathcal{O}_X$-module maps $\mathcal{F} \to \mathcal{G}$.

(e) Let $\phi\colon Y \to X$ be a map of ringed spaces. If $\mathcal{G}$ is a $\mathcal{O}_Y$-module, then $\phi_*\mathcal{G}$ is a $\phi_*\mathcal{O}_Y$-module, and hence can be regarded as a $\mathcal{O}_X$-module thanks to the given map $\phi^*\colon \mathcal{O}_X \to \phi_*\mathcal{O}_Y$. There is also a module pull-back functor $\mathcal{F} \mapsto \phi^*\mathcal{F}$ from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules, defined as follows

$$\phi^*\mathcal{F} = \phi^*_{\mathrm{Ab}}(\mathcal{F}) \otimes_{\phi^*_{\mathrm{Ab}}(\mathcal{O}_X)} \mathcal{O}_Y$$

where ϕ^*_{Ab} denotes pull-back of sheaves abelian groups[3] (see Lecture 7). In other words, introducing the sheaf of rings $\mathcal{O}_{Y/X} = \phi^*_{\mathrm{Ab}}(\mathcal{O}_X))$, the functor is the composition of two functors

$$\mathbf{Mod}_{\mathcal{O}_X} \xrightarrow{\phi^*_{\mathrm{Ab}}} \mathbf{Mod}_{\mathcal{O}_{Y/X}} \xrightarrow{(-)\otimes_{\mathcal{O}_{Y/X}}\mathcal{O}_Y} \mathbf{Mod}_{\mathcal{O}_X}.$$

It is easy to deduce from this presentation that ϕ^* is left adjoint to ϕ_*.

[3] Certain sources (e.g. Hartshorne) use the notation $\phi^{-1}(\mathcal{F})$ for our $\phi^*_{\mathrm{Ab}}(\mathcal{F})$.

Examples 10.1.3. 1. If X is a C^∞-manifold and $\mathcal{O}$ is the sheaf of smooth real-valued functions on X, the sheaf $\mathcal{M}$ of differential 1-forms on X has a natural structure of an $\mathcal{O}$-module. This simply says that we can multiply a differential form by a smooth function.

2. Let $X = \operatorname{Spec}(A)$ be an affine scheme and let M be an A-module. As we shall see directly below, we can turn M into a $\mathcal{O}_X$-module $\tilde{M}$ whose value on a distinguished open $D(f) \subseteq X$ is $M[f^{-1}]$.

Lemma 10.1.4 (compare Corollary 8.2.2)**.** *Let X be either an affine scheme or an affine algebraic set, let $A = \mathcal{O}(X)$, and let M be an A-module. There exists a unique $\mathcal{O}_X$-module $\tilde{M}$ such that for every $f \in A$ we have*

$$\tilde{M}(D(f)) = M[f^{-1}]$$

(and such that for $f, g \in A$, the restriction map $\tilde{M}(D(f)) \to \tilde{M}(D(fg))$ coincides with the natural map $A[f^{-1}] \to A[(fg)^{-1}]$). Moreover, for every $x \in X$ corresponding to a prime ideal $\mathfrak{p}_x \subseteq A$, we have

$$\tilde{M}_x \simeq M_{\mathfrak{p}_x}.$$

Proof. Same as the proof of Corollary 8.2.2, only now we use Lemma 9.1.1 in place of Lemma 8.2.1. □

Remark 10.1.5. A shorter construction of the sheaf $\tilde{M}$ (which works in more general settings). Consider the one-point ringed space $\star_A = (\star, A)$ with structure sheaf given by A. There is an obvious map $p \colon X \to \star_A$. On the other hand, A-modules are $\mathcal{O}$-modules of $\star_A$, and for an A-module M, we have $\tilde{M} = p^* M$.

The above lemma constructs a functor

$$\widetilde{(-)} \colon \mathbf{Mod}_{\mathcal{O}(X)} \to \mathbf{Mod}_{\mathcal{O}_X}.$$

We shall study its properties in the next few lemmas.

Lemma 10.1.6 (Hartshorne II Proposition 5.2(d,e))**.** *Let $\phi \colon Y \to X$ be a map of affine schemes or affine algebraic sets, and let $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Y)$.*

(a) Let M be an A-module. Then $f^*(\tilde{M})$ is naturally isomorphic to $\widetilde{M \otimes_A B}$. In other words, the diagram of functors

$$\begin{array}{ccc} \mathbf{Mod}_A & \xrightarrow{(-)\otimes_A B} & \mathbf{Mod}_B \\ \downarrow & & \downarrow \\ \mathbf{Mod}_{\mathcal{O}_X} & \xrightarrow[\phi^*]{} & \mathbf{Mod}_{\mathcal{O}_Y} \end{array}$$

commutes.

(b) Let N be an A-module. Then $f_*(\tilde{N})$ is naturally isomorphic to $\tilde{N}$ for N treated as an A-module. In other words, the diagram of functors

$$\begin{array}{ccc} \mathbf{Mod}_A & \xleftarrow{forget} & \mathbf{Mod}_B \\ \downarrow & & \downarrow \\ \mathbf{Mod}_{\mathcal{O}_X} & \xleftarrow[\phi_*]{} & \mathbf{Mod}_{\mathcal{O}_Y} \end{array}$$

commutes.

Proof. This is clear once you unwrap the definitions. □

Lemma 10.1.7 (Hartshorne II Proposition 5.2(a,b,c))**.** *Let $X = \operatorname{Spec}(A)$. The functor $M \mapsto \tilde{M}$ from A-modules to $\mathcal{O}_X$-modules is fully faithful, conservative, and exact, and commutes with tensor product and arbitrary direct sums. For every $\mathcal{O}_X$-module $\mathcal{F}$ we have*

$$\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) = \operatorname{Hom}_A(M, \mathcal{F}(X)).$$

Proof. Since $\tilde{M}(X) = M$, the functor is faithful and conservative. Let $f\colon \tilde{M} \to \tilde{N}$ be a map, and let $f'\colon \tilde{M} \to \tilde{N}$ be the map induced by the map $M \to N$ obtained by applying global sections to f. At every point $x \in X$, the stalks of f and f' agree, and hence $f = f'$, which shows fullness. A similar argument involving stalks proves exactness. The assertions about $\otimes$ and $\bigoplus$ are clear from the definition of $\tilde{M}$. The final assertion is easiest to see through the lens of Remark 10.1.5. □

10.2. Quasi-coherent sheaves

Definition 10.2.1. Let X be either a scheme or an algebraic set. A $\mathcal{O}_X$-module $\mathcal{F}$ is **quasi-coherent** if there exists an affine open cover $X = \bigcup U_\alpha$ such that $\mathcal{F}|_{U_\alpha}$ is of the form $\tilde{M}_\alpha$ for some $\mathcal{O}(U_\alpha)$-module M_α. If X is either a locally noetherian scheme or an algebraic set, we say that $\mathcal{F}$ is **coherent** if the same condition holds with each M_α a finitely generated $\mathcal{O}(U_\alpha)$-module.

Example 10.2.2 (A non-quasi-coherent $\mathcal{O}_X$-module)**.** Consider the open immersion

$$j\colon U = \mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1 = X.$$

The extension by zero $j_!\mathcal{O}_U \subseteq \mathcal{O}_X$ is defined as

$$(j_!\mathcal{O}_U)(V) = \{f \in \mathcal{O}_U(U \cap V) : f_x = 0 \text{ for all } x \in V \setminus U\}$$

(see [Hartshorne Ex. II 1.19(b)]). In our case, this means that

$$(j_!\mathcal{O}_U)(V) = \begin{cases} \mathcal{O}(V) & \text{if } V \subseteq U \\ 0 & \text{otherwise.} \end{cases}$$

Then $j_!(\mathcal{O}_U)(U) \neq 0$, so $j_!(\mathcal{O}_U) \neq 0$. On the other hand, we have $j_!(\mathcal{O}_U)(X) = 0$. It follows that $j_!\mathcal{O}_U$ is not quasi-coherent (since it is nonzero but has no global sections). The quotient $\mathcal{O}_X/j_!\mathcal{O}_U$ is the skyscraper at 0 with value $\mathcal{O}_{X,0}$, which is not quasi-coherent either. See also [Har77, Example II 5.2.3] and [Mum99, §III.1, Example A, p. 142].

Remark 10.2.3. Quasi-coherent and coherent sheaves have the following simple characterization, which makes sense on any ringed space[4]. An $\mathcal{O}_X$-module is quasi-coherent if and only if every $x \in X$ has an open neighborhood $U \subseteq X$ on which $\mathcal{F}$ admits a presentation of the form

$$\mathcal{O}_U^{\oplus J} \longrightarrow \mathcal{O}_U^{\oplus I} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

for some sets I and J. Moreover, $\mathcal{F}$ is coherent if and only if the same holds with I and J finite. It follows that coherent sheaves are the smallest class of $\mathcal{O}_U$-modules on opens $U \subseteq X$ such that (a) being coherent is a local condition, (b) $\mathcal{O}_U$ is coherent, (c) coherent sheaves are closed under direct sums and cokernels.

Lemma 10.2.4. *Let $X = \operatorname{Spec}(A)$ be an affine scheme and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the natural map*

$$\widetilde{\mathcal{F}(X)} \longrightarrow \mathcal{F}$$

(the counit of the adjunction in Lemma 10.1.7) is an isomorphism. In particular, $\mathcal{F}$ is of the form $\tilde{M}$.

[4] However, the resulting notion will not always be so well-behaved in such generality.

Proof. Note that since the functor $M \to \tilde{M}$ is fully faithful and the global sections functor is its right adjoint (Lemma 10.1.7), an $\mathcal{O}_X$-module is of the form $\tilde{M}$ if and only if its counit map $\widetilde{\mathcal{F}(X)} \to \mathcal{F}$ is an isomorphism. For brevity, let us call such $\mathcal{O}_X$-modules *in the essential image* (of the functor $M \mapsto \tilde{M}$). By Lemma 10.1.7, being in the essential image is closed under kernels, cokernels, and finite direct sums.

By definition of a quasi-coherent $\mathcal{O}_X$-module, there exists an affine open cover $X = \bigcup_{\alpha \in I} U_\alpha$ such that $\mathcal{F}|_{U_\alpha}$ is of the form $\tilde{M_\alpha}$ for some $\mathcal{O}(U_\alpha)$-module M_α. Since $\mathrm{Spec}(A)$ is quasi-compact, we may assume that the index set I is finite. By Lemma 10.1.6, if $j_\alpha \colon U_\alpha \to X$ is the inclusion, the $\mathcal{O}_X$-module

$$\mathcal{F}_\alpha := j_{\alpha,*} j_\alpha^*(\mathcal{F}) = j_{\alpha,*}(\mathcal{F}|_{U_\alpha}) = j_{\alpha,*}(\tilde{M_\alpha})$$

is equal to $\tilde{M_\alpha}$ where M_α is treated as an A-module (via the forgetful functor $\mathbf{Mod}_{\mathcal{O}(U_\alpha)} \to \mathbf{Mod}_A$), and hence in the essential image. In particular the sheaf

$$\mathcal{F}_0 = \bigoplus_{\alpha \in I} \mathcal{F}_\alpha = \bigoplus_{\alpha \in I} j_{\alpha,*} j_\alpha^*(\mathcal{F})$$

is in the essential image (as this condition is closed under finite direct sums). Since the intersections $U_\alpha \cap U_\beta$ are affine (Lemma 9.4.2), we can apply the same reasoning to the sheaf

$$\mathcal{F}_1 = \bigoplus_{\alpha,\beta \in I} j_{\alpha\beta,*} j_{\alpha\beta}^*(\mathcal{F}), \qquad j_{\alpha\beta} \colon U_\alpha \cap U_\beta \to X.$$

Now, by the sheaf condition for $\mathcal{F}$, the following sequence of $\mathcal{O}_X$-modules is exact

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \xrightarrow{d} \mathcal{F}_1, , \qquad d(s)_{\alpha\beta} = s_\alpha - s_\beta.$$

and hence $\mathcal{F} = \ker(d \colon \mathcal{F}_0 \to \mathcal{F}_1)$ is in the essential image. □