The Riemann-Hilbert correspondence and Fourier transform (I)

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Riemann-Hilbert correspondence

Meromorphic flat bundles are described *in a topological way*, or more roughly, *as tuples of vector spaces and linear maps*.

Regular case

$$\left(\begin{array}{c} \textbf{Regular meromorphic} \\ \textbf{flat bundles on } (X, D) \end{array} \right) \longleftrightarrow \left(\begin{array}{c} \textbf{Local systems} \\ \textbf{on } X \setminus D \end{array} \right)$$

General meromorphic flat bundles

$$\left(\begin{array}{c} \textbf{Meromorphic flat bundles} \\ \textbf{on } (X, D) \end{array} \right) \longleftrightarrow \left(\begin{array}{c} \textbf{Local systems on } X \setminus D \\ \textbf{with Stokes structure at } P \in D \end{array} \right)$$

Remark **More recently,** *D'Agnolo, Kashiwara and Schapira* **developed the theory of enhanced ind-sheaves, and proved that the category of holonomic $\mathscr{D}$-modules is functorially embedded into the category of $\mathbb{R}$-constructible enhanced ind-sheaves.**

A general problem

Problem *For a given integral functor F, how G is described, or explicitly computed?*

$$\begin{array}{ccc}
\big(\,\textbf{Mero. flat bundles on } (X, D_X)\,\big) & \xrightarrow{\ F\ } & \big(\,\textbf{Mero. flat bundles on } (Y, D_Y)\,\big) \\
\simeq\big\downarrow & & \simeq\big\downarrow \\
\big(\,\textbf{Stokes-Loc. sys. on } (X, D_X)\,\big) & \xrightarrow{\ G?\ } & \big(\,\textbf{Stokes-Loc. sys. on } (Y, D_Y)\,\big)
\end{array}$$

Let K be a holonomic $\mathscr{D}$-module on $X \times Y$.
Let $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ be the projections.
Suppose that F is given as

$$F(M) := p_{2+}\big(p_1^*(M) \otimes K\big)$$

Abstract answers

- *translate F into a functor for $\mathbb{R}$-constr. enhanced ind-sheaves.*
- *more direct approach using only constructible sheaves.*

They are not so easy to compute.

Problem' *For a given integral functor F, how G' is described, or explicitly computed?*

$$\begin{array}{ccc}
\left(\textbf{ Mero. flat bundles on } (X, D_X)\right) & \xrightarrow{F} & \left(\textbf{ Mero. flat bundles on } (Y, D_Y)\right) \\
\simeq\big\downarrow & & \simeq\big\downarrow \\
\left(\begin{array}{c}\textbf{tuples of vector spaces} \\ \textbf{and linear maps}\end{array}\right) & \xrightarrow{G'\,?} & \left(\begin{array}{c}\textbf{tuples of vector spaces} \\ \textbf{and linear maps}\end{array}\right)
\end{array}$$

Goal of this talk

$$\mathfrak{Four} \curvearrowright \left(\begin{array}{c} \textbf{algebraic holonomic} \\ \mathscr{D}\textbf{-modules on } \mathbb{C} \end{array} \right) \quad \textbf{(Fourier transform)}$$

We study the Stokes structure of $\mathfrak{Four}(M)$ at ∞.

- **Introduce another way to formulate Stokes structures**
  (Stokes shells)
- **Describe the Stokes structure of $\mathfrak{Four}(M)$ at ∞.**

Plan

- **Riemann-Hilbert correspondence (1-dim)**
- **Fourier transform**
- **Main results (a description of the Stokes structure of $\mathfrak{Four}(M)$ at ∞)**
- **Stokes shells**

Riemann-Hilbert correspondence (1-dim)

Meromorphic flat bundles on a punctured disc

Let $\Delta := \{z \in \mathbb{C} \mid |z| < 1\}$. Let $\mathscr{O}_\Delta(*0)$ be the sheaf of meromorphic functions which may have pole along 0.

Let $\mathscr{V}$ be a locally free $\mathscr{O}_\Delta(*0)$-module. A connection $\nabla : \mathscr{V} \longrightarrow \mathscr{V} \otimes \Omega^1$ is a differential operator such that $\nabla(fs) = f\nabla(s) + s \otimes df$. Such $(\mathscr{V}, \nabla)$ is called *a meromorphic flat bundle* **on $(\Delta, 0)$.**

$(\mathscr{V}, \nabla)$ is called *regular* **if there exists a locally free $\mathscr{O}_\Delta$-submodule $V \subset \mathscr{V}$ such that**

$$V \otimes \mathscr{O}_\Delta(*0) = \mathscr{V}, \quad \nabla(V) \subset V \otimes \Omega^1(0).$$

$(\mathscr{V}, \nabla)$ is called *irregular* **if it is not regular.**

$$\begin{array}{rl} \mathrm{MF}(\Delta, 0) & := \Big(\textbf{meromorphic flat bundles on } (\Delta, 0)\Big) \\ \bigcup & \\ \mathrm{MF}^{\mathrm{reg}}(\Delta, 0) & := \Big(\textbf{regular singular}\Big) \end{array}$$

The associated local systems

Let $(\mathscr{V},\nabla)$ be a meromorphic flat bundle on $(\Delta,0)$. For $\mathscr{U}\subset\Delta^*$,

$$\mathrm{Loc}(\mathscr{V},\nabla)(\mathscr{U}):=\bigl\{s\in\mathscr{V}(\mathscr{U})\bigm|\nabla(s)=0\bigr\}.$$

We obtain the local system $\mathrm{Loc}(\mathscr{V},\nabla)$ on $\Delta^*=\Delta\setminus\{0\}$ of flat sections.

Theorem (RH-correspondence in the regular case)

$(\mathscr{V},\nabla)\longmapsto\mathrm{Loc}(\mathscr{V})$ **induces**

$$\mathrm{MF}^{\mathrm{reg}}(\Delta,0)\simeq\mathrm{Loc}(\Delta,0):=\begin{pmatrix}\textbf{Local systems}\\ \textbf{on }\Delta^*\end{pmatrix}$$

$\mathrm{MF}(\Delta,0) \longrightarrow \mathrm{Loc}(\Delta,0)$ **is not equivalent.**

Example

For $\mathfrak{a} \in z^{-1}\mathbb{C}[z^{-1}]$**, we have the meromorphic flat bundle**

$$\mathscr{L}_{\mathfrak{a}} := (\mathscr{O}_{\Delta}(*0), d + d\,\mathfrak{a}).$$

If $\mathfrak{a}_1 \neq \mathfrak{a}_2$**, then** $\mathscr{L}_{\mathfrak{a}_1} \not\simeq \mathscr{L}_{\mathfrak{a}_2}$ **as meromorphic flat bundles. But,** $\mathrm{Loc}(\mathscr{L}_{\mathfrak{a}_i})$ **are isomorphic to the constant sheaf** $\mathbb{C}_{\Delta^*}$**. (**$\exists$ **a global section** $\exp(-\mathfrak{a}_i)$**.)**

Meromorphic flat bundles on formal punctured disc

Let $\widehat{\mathscr{V}}$ be a $\mathbb{C}((z))$-vector space. A connection on $\widehat{\mathscr{V}}$ is a $\mathbb{C}$-linear map $\nabla : \widehat{\mathscr{V}} \longrightarrow \Omega^1_{\mathbb{C}((z))/\mathbb{C}} \otimes \widehat{\mathscr{V}}$ such that $\nabla(fs) = f\nabla(s) + df \otimes s$.
The regularity is defined similarly.

$$\begin{array}{cl} \mathrm{MF}(\mathbb{C}((z))) & := \Big(\textbf{meromorphic flat bundles on } \mathbb{C}((z))\Big) \\ \bigcup & \\ \mathrm{MF}^{\mathrm{reg}}(\mathbb{C}((z))) & := \Big(\textbf{regular singular}\Big) \end{array}$$

Formal classification

Hukuhara-Levelt-Turrittin theorem

For any $(\widehat{\mathscr{V}},\nabla)\in \mathrm{MF}(\mathbb{C}((z)))$**, there exist** $m\in\mathbb{Z}_{>0}$ **and a decomposition**

$$(\widehat{\mathscr{V}}\otimes\mathbb{C}((z^{1/m})),\nabla)=\bigoplus_{\mathfrak{a}\in z^{-1/m}\mathbb{C}[z^{-1/m}]}(\widehat{\mathscr{R}}_{\mathfrak{a}},\nabla_{\mathfrak{a}})\otimes\widehat{\mathscr{L}_{\mathfrak{a}}},$$

where $(\widehat{\mathscr{R}}_{\mathfrak{a}},\nabla_{\mathfrak{a}})$ **are regular, and** $\widehat{\mathscr{L}_{\mathfrak{a}}}:=(\mathbb{C}((z^{1/m})),d+d\mathfrak{a})$**.**
Put $\mathscr{I}(\widehat{\mathscr{V}}):=\{\mathfrak{a}\,|\,\widehat{\mathscr{R}}_{\mathfrak{a}}\neq 0\}\subset z^{-1/m}\mathbb{C}[z^{-1/m}]$**.**
It is invariant under the natural action of $\mathrm{Gal}(m):=\{a\in\mathbb{C}\,|\,a^m=1\}$**.**

Notation **For any** $\mathrm{Gal}(m)$**-invariant subset** $\mathscr{I}\subset z^{-1/m}\mathbb{C}[z^{-1/m}]$**,**

$$\mathrm{MF}(\mathbb{C}((z)))\supset\mathrm{MF}(\mathbb{C}((z));\mathscr{I}):=\big((\widehat{\mathscr{V}},\nabla)\in\mathrm{MF}(\mathbb{C}((z))),\ \mathscr{I}(\widehat{\mathscr{V}})\subset\mathscr{I}\big).$$

Formal completion

$$\mathrm{MF}(\Delta,0) \ni (\mathscr{V},\nabla) \longmapsto (\mathscr{V},\nabla)_{|\widehat{0}} := (\mathscr{V} \otimes_{\mathscr{O}} \mathbb{C}[\![z]\!],\nabla) \in \mathrm{MF}(\mathbb{C}(\!(z)\!))$$

- $\mathrm{MF}^{\mathrm{reg}}(\Delta,0) \simeq \mathrm{MF}^{\mathrm{reg}}(\mathbb{C}(\!(z)\!))$.
- $\mathrm{MF}(\Delta,0) \not\simeq \mathrm{MF}(\mathbb{C}(\!(z)\!))$ *(not fully faithful)*

  Formal isomorphisms are not necessarily convergent.

Notation

For any $(\mathscr{V},\nabla) \in \mathrm{MF}(\Delta,0)$**, we set** $\mathscr{I}(\mathscr{V}) := \mathscr{I}(\mathscr{V}_{|\widehat{0}})$.

For any $\mathrm{Gal}(m)$**-invariant subset** $\mathscr{I} \subset z^{-1/m}\mathbb{C}[z^{-1/m}]$,

$$\mathrm{MF}(\Delta,0) \supset \mathrm{MF}(\Delta,0;\mathscr{I}) := \big((\mathscr{V},\nabla) \in \mathrm{MF}(\Delta,0),\ \mathscr{I}(\mathscr{V}) \subset \mathscr{I}\big).$$

Stokes structures

We need to consider *Stokes structure* **to obtain an equivalence of meromorphic flat bundles and some topological objects. There are several formulations.**

Deligne **and** *Malgrange* **formulated it as** *a family of Stokes filtrations*.

Remark **We shall naturally identify**

$$\mathrm{Loc}(\Delta, 0) \simeq \begin{pmatrix} 2\pi\mathbb{Z}\text{-equivariant} \\ \text{local systems on } \mathbb{R} \end{pmatrix}, \quad \begin{cases} \mathscr{L} \longmapsto \varphi^*(\mathscr{L}), \\ (\varphi(\theta) = \varepsilon e^{\sqrt{-1}\theta}) \end{cases}$$

Partial orders depending on the direction

For $\mathfrak{a} = \mathfrak{a}_\omega z^{-\omega} + \sum_{0<\gamma<\omega} \mathfrak{a}_\gamma z^{-\gamma} \in z^{-1/m}\mathbb{C}[z^{-1/m}]$ with $\mathfrak{a}_\omega \neq 0$, we obtain the function $G_{\mathfrak{a}}$ on $\mathbb{R}$:

$$G_{\mathfrak{a}}(\theta) := -\frac{\mathrm{Re}\big(\mathfrak{a}_\omega e^{-\omega\sqrt{-1}\theta}\big)}{|\mathfrak{a}_\omega|}.$$

$$\begin{aligned} G_{\mathfrak{a}}(\theta_0) < 0 \quad &\Longleftrightarrow -\mathrm{Re}(\mathfrak{a}(re^{\sqrt{-1}\theta})) < 0 \text{ for } |\theta-\theta_0| << 1 \text{ and } 0 < r << 1 \\ &\Longleftrightarrow |e^{-\mathfrak{a}}| \text{ decays rapidly on the sector } |\theta-\theta_0| << 1. \end{aligned}$$

The order $\leq_\theta$ **on $z^{-1/m}\mathbb{C}[z^{-1/m}]$ is defined by**

$$\mathfrak{a} \leq_\theta \mathfrak{b} \Longleftrightarrow \mathfrak{a} = \mathfrak{b}, \text{or } G_{\mathfrak{a}-\mathfrak{b}}(\theta) < 0.$$

For any $\mathfrak{a} \neq \mathfrak{b} \in z^{-1/m}\mathbb{C}[z^{-1/m}]$, we set $\mathrm{St}(\mathfrak{a},\mathfrak{b}) := \{\theta \in \mathbb{R} \,|\, G_{\mathfrak{a}-\mathfrak{b}}(\theta) = 0\}$.

- **$\leq_\theta$ on $\{\mathfrak{a},\mathfrak{b}\}$ is constant on any connected component of $\mathbb{R} \setminus \mathrm{St}(\mathfrak{a},\mathfrak{b})$.**
- **$\mathfrak{a} \not\leq_\theta \mathfrak{b}$, $\mathfrak{a} \not\geq_\theta \mathfrak{b}$ at $\theta \in \mathrm{St}(\mathfrak{a},\mathfrak{b})$.**
- **$\mathfrak{a} \leq_{\theta_0} \mathfrak{b} \Longrightarrow \mathfrak{a} \leq_{\theta_1} \mathfrak{b}$ (if θ_1 is sufficiently close to θ_0).**
- **$\leq_\theta$ is changed when θ goes through any point of $\mathrm{St}(\mathfrak{a},\mathfrak{b})$.**

Stokes structure on local systems

Set $\mathrm{Gal}(m) := \{a \in \mathbb{C} \mid a^m = 1\}$ **which naturally acts on** $z^{-1/m}\mathbb{C}[z^{-1/m}]$.
Take any $\mathrm{Gal}(m)$**-invariant finite subset** $\mathscr{I} \subset z^{-1/m}\mathbb{C}[z^{-1/m}]$. $2\pi\mathbb{Z} \curvearrowright \mathscr{I}$.

$$(2\pi k)^* z^{-1/m} = z^{-1/m} e^{-\sqrt{-1}2\pi k/m}, \quad (2\pi k)^*\mathfrak{a} \leq_\theta (2\pi k)^*\mathfrak{b} \iff \mathfrak{a} \leq_{\theta+2\pi k} \mathfrak{b}$$

Let L **be a** $2\pi\mathbb{Z}$**-equivariant local system on** $\mathbb{R}$.

Definition **A** *$2\pi\mathbb{Z}$-equivariant Stokes structure* **on** L **over** $\mathscr{I}$ **is a** $2\pi\mathbb{Z}$**-equivariant family of filtrations** $\boldsymbol{\mathscr{F}} = \left(\mathscr{F}^\theta \mid \theta \in \mathbb{R}\right)$ **on** L_θ **indexed by** $(\mathscr{I}, \leq_\theta)$ **satisfying the condition:**

- $\exists$ **decomposition** $L_\theta = \bigoplus_{\mathfrak{a}\in\mathscr{I}} G_{\theta,\mathfrak{a}}$ **such that** $\mathscr{F}^\theta_{\mathfrak{a}} = \sum_{\mathfrak{b}\leq_\theta\mathfrak{a}} G_{\theta,\mathfrak{b}}$.
- **If** θ_1 **is sufficiently close to** θ_0

$$\mathscr{F}^{\theta_0}_{\mathfrak{a}} \subset \mathscr{F}^{\theta_1}_{\mathfrak{a}} \text{ and } \mathrm{Gr}^{\mathscr{F}^{\theta_0}}_{\mathfrak{a}} \simeq \mathrm{Gr}^{\mathscr{F}^{\theta_1}}_{\mathfrak{a}} \quad \left(\mathrm{Gr}^{\mathscr{F}^\theta}_{\mathfrak{a}} = \mathscr{F}^\theta_{\mathfrak{a}} \Big/ \sum_{\mathfrak{b}<_\theta\mathfrak{a}} \mathscr{F}^\theta_{\mathfrak{b}}\right)$$

Set $\mathrm{St}(\mathscr{I}) := \bigcup_{\mathfrak{a}\neq\mathfrak{b}\in\mathscr{I}} \mathrm{St}(\mathfrak{a},\mathfrak{b}) = \bigcup_{\mathfrak{a}\neq\mathfrak{b}\in\mathscr{I}} \{G_{\mathfrak{a}-\mathfrak{b}}(\theta) = 0\}$.

- $\mathscr{F}^\theta$ **are constant on any connected component of** $\mathbb{R}\setminus\mathrm{St}(\mathscr{I})$.
- $\mathscr{F}^\theta$ **is changed when** θ **goes through any points of** $\mathrm{St}(\mathscr{I})$.

Morphisms

$$f:(L_1,\mathscr{F})\longrightarrow(L_2,\mathscr{F})\ \textbf{morphism} \overset{\text{def}}{\Longleftrightarrow} \begin{cases} f:L_1\longrightarrow L_2\ \textbf{morphism(equivariant)} \\ f(\mathscr{F}^{\theta}_{\mathfrak{a}}L_{1,\theta})\subset\mathscr{F}^{\theta}_{\mathfrak{a}}L_{2,\theta}\ (\forall\theta,\mathfrak{a}) \end{cases}$$

Remark **Indeed,** $f(\mathscr{F}^{\theta}_{\mathfrak{a}}L_{1,\theta})=f(L_{1,\theta})\cap\mathscr{F}^{\theta}_{\mathfrak{a}}(L_{2,\theta})$ **(strict).**

Notation

$$\mathrm{Loc}^{\mathrm{St}}:=\left(\begin{array}{c}2\pi\mathbb{Z}\textbf{-equivariant local systems}\\ \textbf{with Stokes structure on } \mathbb{R}\end{array}\right)$$

$$\mathrm{Loc}^{\mathrm{St}}(\mathscr{I}):=\left(\begin{array}{c}2\pi\mathbb{Z}\textbf{-equivariant local systems}\\ \textbf{with Stokes structure on } \mathbb{R} \textbf{ over } \mathscr{I}\end{array}\right)$$

Remark **Later, we shall also consider** $2\pi m\mathbb{Z}$**-equivariant versions for** $m\in\mathbb{Z}_{>0}$**. We use "**$\mathrm{Loc}^{\mathrm{St}}_m$**" and "**$\mathrm{Loc}^{\mathrm{St}}_m(\mathscr{I})$**".**

Riemann-Hilbert correspondence

Theorem (Deligne, Malgrange)

$$\mathrm{MF}(\Delta, 0; \mathscr{I}) \simeq \mathrm{Loc}^{\mathrm{St}}(\mathscr{I})$$

Stokes filtrations describe the growth orders of flat sections in the direction θ.

For a small $\varepsilon > 0$, let $\varphi : \mathbb{R} \longrightarrow \Delta \setminus \{0\}$ be given by $\varphi(\theta) = \varepsilon e^{\sqrt{-1}\theta}$.
For $(\mathscr{V}, \nabla) \in \mathrm{MF}(\Delta, 0; \mathscr{I})$, we obtain $L(\mathscr{V}) := \varphi^* \mathrm{Loc}(\mathscr{V}, \nabla)$:

$$s \in \mathscr{F}^{\theta}_{\mathfrak{a}} L(\mathscr{V})_{\theta} \Longleftrightarrow \left|\exp(\mathfrak{a}) s\right| = O\big(|z|^{-N}\big) \text{ on a sector around } \arg(z) = \theta \quad (\exists N),$$

(More precisely, $s = \sum s_j$ for a frame $(v_1, \ldots, v_r)$ of $\mathscr{V}$, then $|\exp(\mathfrak{a}) s_j| = O\big(|z|^{-N}\big)$.)

Easy examples

- **Let** $(\mathscr{V},\nabla)\in \mathrm{MF}^{\mathrm{reg}}(\Delta,0)$**.**

$$\mathscr{F}^{\theta}_{\mathfrak{a}}L(\mathscr{V},\nabla)_{\theta}=\begin{cases} L(\mathscr{V},\nabla)_{\theta} & (\mathfrak{a}\geq_{\theta}0)\\ 0 & (\text{otherwise})\end{cases}$$

- $\mathscr{L}_{\mathfrak{b}}=(\mathscr{O}_{\Delta}(*0),d+d\mathfrak{b})$**. Then,** $\mathrm{Loc}(\mathscr{L}_{\mathfrak{b}})=\mathbb{C}\cdot\exp(-\mathfrak{b})$**, and**

$$\mathscr{F}^{\theta}_{\mathfrak{a}}L(\mathscr{L}_{\mathfrak{b}})_{\theta}=\begin{cases} L(\mathscr{L}_{\mathfrak{b}})_{\theta} & (\mathfrak{a}\geq_{\theta}\mathfrak{b})\\ 0 & (\textbf{otherwise})\end{cases}$$

We may distinguish $\mathscr{L}_{\mathfrak{b}_1}$ **and** $\mathscr{L}_{\mathfrak{b}_2}$ $(\mathfrak{b}_1\neq\mathfrak{b}_2)$ **by using the Stokes filtrations.**

Remark **In general, it is difficult to compute** $(L(\mathscr{V},\nabla),\mathscr{F})$ **from** $(\mathscr{V},\nabla)$**.**

$\mathrm{Gr}^{\mathscr{F}}$ and the formal completion

For $(L,\mathscr{F})\in\mathrm{Loc}^{\mathrm{St}}(\mathscr{I})$**, the vector spaces** $\mathrm{Gr}_{\mathfrak{a}}^{\mathscr{F}^\theta}(L_\theta)$ **(**$\mathfrak{a}\in\mathscr{I}$**,** $\theta\in\mathbb{R}$**) naturally induce a local system** $\mathrm{Gr}_{\mathfrak{a}}^{\mathscr{F}}(L)$ **on** $\mathbb{R}$**.**

The direct sum $\mathrm{Gr}^{\mathscr{F}}(L):=\bigoplus_{\mathfrak{a}\in\mathscr{I}}\mathrm{Gr}_{\mathfrak{a}}^{\mathscr{F}}(L)$ **is naturally** $2\pi\mathbb{Z}$**-equivariant, and equipped with the induced Stokes structure** $\mathscr{F}$**.**

$$\mathscr{F}_{\mathfrak{b}}^{\theta}\,\mathrm{Gr}^{\mathscr{F}}(L)_\theta=\bigoplus_{\mathfrak{a}\leq_\theta\mathfrak{b}}\mathrm{Gr}_{\mathfrak{a}}^{\mathscr{F}}(L)_\theta.$$

When $(\mathscr{V},\nabla)\longleftrightarrow(L,\mathscr{F})$ **and** $(\mathscr{V}_{|\widehat{0}}\otimes\mathbb{C}((z^{-1/m})),\nabla)\simeq\bigoplus(\mathscr{L}_{\mathfrak{a}}\otimes\mathscr{R}_{\mathfrak{a}})_{|\widehat{0}}$**,**

$$\begin{array}{ccc}(\mathscr{V},\nabla) & \Longrightarrow & \textbf{descent of } \bigoplus\mathscr{L}_{\mathfrak{a}}\otimes\mathscr{R}_{\mathfrak{a}}\\ \updownarrow & & \updownarrow\\ (L,\mathscr{F}) & \Longrightarrow & (\mathrm{Gr}^{\mathscr{F}}(L),\mathscr{F}).\end{array}$$

Globalization

Let X be a complex curve with a discrete subset D. Let (X_P, z_P) $(P \in D)$ be coordinate neighbourhoods.

Meromorphic flat bundles on (X, D) are equivalent to

- **local systems $\mathscr{L}$ on $X \setminus D$**

  (we obtain $2\pi\mathbb{Z}$-equivariant local systems L_P $(P \in D)$ on $\mathbb{R}$ from $\mathscr{L}_{|X_P \setminus \{P\}}$)

- **$2\pi\mathbb{Z}$-equivariant local systems with Stokes structure $(L_P, \mathscr{F}_P)$ $(P \in D)$**

Fourier transform

Algebraic $\mathscr{D}$-modules on $\mathbb{C}$ are equivalent to modules over the Weyl algebra $\mathbb{C}[z]\langle\partial_z\rangle$. We have the automorphism of the Weyl algebra given by

$$(\partial_z, z) \longmapsto (-z, \partial_z).$$

Fourier transform is the induced auto-equivalence on the category of modules over the Weyl algebra, or on the category of algebraic $\mathscr{D}_{\mathbb{C}}$-modules.

Let $p_i : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ be the projection onto the i-th component.
Set $\mathscr{L}(zw) = (\mathscr{O}_{\mathbb{C}\times\mathbb{C}}, d + d(zw))$. Then, $\mathfrak{Four}(M) = p_{1+}\big(p_2^*(M) \otimes \mathscr{L}(zw)\big)$.

Question

It is natural to ask

$$\begin{pmatrix} \textbf{Local sys. with} \\ \textbf{Stokes structure} \\ \textbf{(+ something)} \\ \textbf{corresponding to } M \end{pmatrix} \xrightarrow{how?} \begin{pmatrix} \textbf{Local sys. with} \\ \textbf{Stokes structure} \\ \textbf{(+ something)} \\ \textbf{corresponding to } \mathfrak{Four}(M) \end{pmatrix}$$

(Arinkin, Beilinson, Bloch, Deligne, D'Agnolo, Esnault, Fang, Fu, Graham-Squire, Hien, Kashiwara, Laumon, Malgrange, Sabbah, M,.....)

If M is holonomic, then $\mathfrak{Four}(M)_{|U_\infty} \in \mathrm{MF}(U_\infty, \infty)$.

> *Question*
>
> **In particular, how to compute $(\mathfrak{L}^{\mathfrak{F}}(M), \mathscr{F}) \in \mathrm{Loc}^{\mathrm{St}}$ corresponding to $\mathfrak{Four}(M)_{|U_\infty}$?**

Remark **The construction of Fourier transform can be translated to the context of constructible sheaves on the real blow ups or enhanced ind-sheaves. We would like to obtain a more computable way.**

Regular singular case (Malgrange)

Assume M is regular singular at $\mathrm{Sing}(M)\cup\{\infty\}$.

- $\mathscr{I}(\mathfrak{Four}(M),\infty)=\{\beta u^{-1}\mid \beta\in\mathrm{Sing}(M)\}$ **(where $u=w^{-1}$)**

$$\mathfrak{Four}(M)_{|\widehat{\infty}}=\bigoplus_{\beta\in\mathrm{Sing}(M)}\mathscr{L}_{\beta w}\otimes\begin{pmatrix}\text{reg. sing. mero. flat bundle}\\ \text{determined by } \psi_{z-\beta}(M)\leftrightarrows\phi_{z-\beta}(M)\end{pmatrix}.$$

  It determines $\mathrm{Gr}^{\mathscr{F}}(\mathfrak{L}^{\mathfrak{F}}(M))$.

- **Moreover, $(\mathfrak{L}^{\mathfrak{F}}(M),\mathscr{F})$ is computed in terms of the monodromy of M.**

Formal completion at ∞

$$\mathfrak{Four}(M)_{|\widehat{\infty}} = \bigoplus_{\beta \in \mathrm{Sing}(M) \setminus \{\infty\}} \Big(\mathfrak{F}^{(0,\infty)}(M_{|\widehat{\beta}}) \otimes \mathscr{L}_{\beta w} \Big) \oplus \mathfrak{F}^{(\infty,\infty)}(M_{|\widehat{\infty}})$$

by the local Fourier transforms *(Bloch-Esnault, Sabbah)*

$$\mathfrak{F}^{(0,\infty)} : \begin{pmatrix} \textbf{holonomic} \\ D\textbf{-modules} \\ \textbf{on } \mathbb{C}((z)) \end{pmatrix} \longrightarrow \mathrm{MF}\big(\mathbb{C}((w^{-1}))\big)$$

$$\mathfrak{F}^{(\infty,\infty)} : \mathrm{MF}\big(\mathbb{C}((z^{-1}))\big) \longrightarrow \mathrm{MF}\big(\mathbb{C}((w^{-1}))\big)$$

Explicit expressions of local Fourier transforms (Sabbah, Laumon, Fu)

Take any non-zero $\rho \in U\mathbb{C}[\![U]\!]$**, and** $\mathfrak{a} \in \mathbb{C}((U))$**. Set** $p := \operatorname{ord}(\rho)$ **and** $n := -\operatorname{ord}(\mathfrak{a})$**. Set**

$$V := \rho_*\big(\mathbb{C}((U)), d + d\mathfrak{a}\big) \otimes R,$$

where $R \in \mathrm{MF}^{\mathrm{reg}}(\mathbb{C}((z)))$ **and** $\rho_* : \mathrm{MF}(\mathbb{C}((U))) \longrightarrow \mathrm{MF}(\mathbb{C}((z)))$ **is induced by** $z = \rho(U)$**.**

$\mathfrak{F}^{(0,\infty)}$ **Set** $\widehat{\rho}^{(0)}(U) := -\dfrac{\rho'(U)}{\mathfrak{a}'(U)}$, $\widehat{\mathfrak{a}}^{(0)}_{\pm}(U) := \mathfrak{a}(U) - \dfrac{\rho(U)}{\rho'(U)}\mathfrak{a}'(U)$**. Then,**

$$\mathfrak{F}^{(0,\infty)}(V) \simeq \widehat{\rho}^{(0)}_*\big(\mathbb{C}((U)), d + d\widehat{\mathfrak{a}}^{(0)} + (n/2)dU/U\big) \otimes R.$$

$\mathfrak{F}^{(\infty,\infty)}$ $(n > p)$ **Set** $\widehat{\rho}^{(\infty)}(U) := \dfrac{\rho'(U)}{\mathfrak{a}(U)\rho(U)^2}$**,** $\widehat{\mathfrak{a}}^{(\infty)}(U) := \mathfrak{a}(U) + \dfrac{\rho(U)}{\rho'(U)}\mathfrak{a}'(U)$**. Then,**

$$\mathfrak{F}^{(\infty,\infty)}(V) \simeq \widehat{\rho}^{(\infty)}_*\big(\mathbb{C}((U)), d + d\widehat{\mathfrak{a}}^{(\infty)}_{\pm} + (n/2)dU/U\big).$$

As a result, $\mathrm{Gr}^{\mathscr{F}}(\mathfrak{L}^{\mathfrak{F}}(M))$ *is explicitly described. We may still ask how to compute* $(\mathfrak{L}^{\mathfrak{F}}(M), \mathscr{F})$*.*

Results on graded pieces

Induced filtrations

Let V be a finite dimensional vector space.

Let $(I, \leq)$ be a finite partially ordered set.

A filtration $\mathscr{F}$ of V indexed by $(I, \leq)$ is a family of vector subspaces $\mathscr{F}_a(V)$ $(a \in I)$ such that (i) $\exists\ V = \bigoplus_{a \in I} G_a$ (ii) $\mathscr{F}_a(V) = \bigoplus_{b \leq a} G_b$. We set

$$\mathscr{F}_{<a}(V) := \sum_{b<a} \mathscr{F}_b(V), \quad \mathrm{Gr}_a^{\mathscr{F}}(V) := \mathscr{F}_a(V) \big/ \mathscr{F}_{<a}(V), \quad \mathrm{Gr}^{\mathscr{F}}(V) := \bigoplus \mathrm{Gr}_a^{\mathscr{F}}(V).$$

Let $\varphi : (I, \leq) \longrightarrow (J, \leq)$ be a morphism of ordered sets.

- We set $(\varphi_* \mathscr{F})_c(V) := \sum_{\varphi(a) \leq c} \mathscr{F}_a(V)$. Thus, we obtain a filtration $\varphi_* \mathscr{F}$ indexed by $(J, \leq)$.
- We obtain a filtration $\mathscr{F}$ on $\mathrm{Gr}_b^{\varphi_* \mathscr{F}}(V)$ indexed by $(\varphi^{-1}(b), \leq)$.

$$\mathscr{F} \text{ on } V \longleftrightarrow \begin{cases} \varphi_*(\mathscr{F}) \text{ on } V \\ \mathscr{F} \text{ on } \mathrm{Gr}_b^{\varphi_* \mathscr{F}}(V) \ (b \in J) \end{cases}$$

Induced local systems with Stokes structure (1)

Let $\omega = \ell/m$ $(\ell, m \in \mathbb{Z}_{>0})$.

Let $\pi_\omega : z^{-1/m}\mathbb{C}[z^{-1/m}] \longrightarrow z^{-\omega}\mathbb{C}[z^{-1/m}]$ **denote the projections:**

$$\pi_\omega\Big(\sum \alpha_a z^{-a}\Big) = \sum_{a \geq \omega} \alpha_j z^{-a}$$

Let $\mathscr{I}$ **be a** $\mathrm{Gal}(m)$**-invariant finite subset of** $z^{-1/m}\mathbb{C}[z^{-1/m}]$.

Let $(L, \boldsymbol{\mathscr{F}}) \in \mathrm{Loc}^{\mathrm{St}}(\mathscr{I})$.

- $\boldsymbol{\mathscr{F}}^{(\omega)} = \big(\mathscr{F}^{(\omega)\,\theta} := \pi_{\omega *}\mathscr{F}^{\theta} \,\big|\, \theta \in \mathbb{R}\big)$ **is a** $2\pi\mathbb{Z}$**-equivariant Stokes structure of** L **over** $\mathscr{I}^{(\omega)} = \pi_\omega(\mathscr{I})$.
- **We obtain the associate graded**

$$\bigoplus_{\mathfrak{b} \in \mathscr{I}^{(\omega)}} \mathrm{Gr}^{(\omega)}_{\mathfrak{b}}(L) = \bigoplus_{\mathfrak{b} \in \mathscr{I}^{(\omega)}} \big(\mathrm{Gr}^{\boldsymbol{\mathscr{F}}^{(\omega)}}_{\mathfrak{b}}(L), \boldsymbol{\mathscr{F}}\big).$$

  Each $\big(\mathrm{Gr}^{(\omega)}_{\mathfrak{b}}(L), \boldsymbol{\mathscr{F}}\big)$ **is a** $2\pi m\mathbb{Z}$**-equivariant local system with Stokes structure over**

$$\Big\{\mathfrak{b} + \sum_{a < \omega} \gamma_a z^{-a} \in \mathscr{I}\Big\}.$$

- $\big(\mathrm{Gr}^{(\omega)}_{0}(L), \boldsymbol{\mathscr{F}}\big)$ **is** $2\pi\mathbb{Z}$**-equivariant.**

Results on graded pieces

Let $\mathscr{V} \in \mathrm{MF}(\mathbb{P}^1, D \cup \{\infty\})$**. Set** $\mathscr{I}_\infty^{\mathfrak{F}} := \mathscr{I}(\mathfrak{Four}(\mathscr{V})_{|\widehat{\infty}})$**.**

Let $\upsilon = \ell/m \in \mathbb{Q}_{>0}$**. Suppose**

$$0 \neq \beta w^\upsilon \in \pi_\upsilon\left(\mathscr{I}_\infty^{\mathfrak{F}}\right).$$

We obtain $2\pi m$**-equivariant** $\mathrm{Gr}^{(\upsilon)}_{\beta w^\upsilon}\left(\mathfrak{L}^{\mathfrak{F}}(\mathscr{V}), \mathscr{F}\right) \in \mathrm{Loc}^{\mathrm{St}}_m$ **over**

$$\mathscr{I}_\infty^{\mathfrak{F}}(\beta w^\upsilon) := \left\{ \beta w^\upsilon + \sum_{0<a<\upsilon} \gamma_a w^a \in \mathscr{I}_\infty^{\mathfrak{F}} \right\}.$$

Claim **For** $\upsilon \neq 1$**,** $\mathrm{Gr}^{(\upsilon)}_{\beta w^\upsilon}\left(\mathfrak{L}^{\mathfrak{F}}(\mathscr{V}), \mathscr{F}\right) \in \mathrm{Loc}^{\mathrm{St}}_m$ **are easily described.**

$\upsilon > 1$

Let $\mathscr{I}_\infty$ be the index set of the HLT decomposition of $\mathscr{V}$ at ∞.
We obtain $(L(\mathscr{V},\infty),\boldsymbol{\mathscr{F}}) \in \mathrm{Loc}^{\mathrm{St}}(\mathscr{I}_\infty)$ from $\mathscr{V}_{|U_\infty} \in \mathrm{MF}(U_\infty,\infty)$.
Let $\mathbb{I}_{m,\ell}$ be $2\pi m\mathbb{Z}$-equivariant local system on $\mathbb{R}$ with a globalization $e_{m,\ell}$ such that $(2\pi m)^* e_{m,\ell} = (-1)^\ell e_{m,\ell}$.

Theorem

There exist $\beta_1 z^\omega \neq 0$ $(\omega = \upsilon(\upsilon-1)^{-1} > 0)$, a bijection

$$\nu^* : \mathscr{I}_\infty(\beta_1 z^\omega) = \left\{ \beta_1 z^\omega + \sum_{0<a<\omega} \gamma_a z^a \in \mathscr{I}_\infty \right\} \simeq \mathscr{I}_\infty^{\mathfrak{F}}(\beta w^\upsilon),$$

and an affine isomorphism $\nu : \mathbb{R} \simeq \mathbb{R}$ such that

- $\nu^* : \left(\mathscr{I}_\infty(\beta_1 z^\omega), \leq_{\nu(\theta)}\right) \simeq \left(\mathscr{I}_\infty^{\mathfrak{F}}(\beta w^\upsilon), \leq_\theta\right)$ **for any $\theta \in \mathbb{R}$,**
- $\mathrm{Gr}^{(\upsilon)}_{\beta w^\upsilon}(\mathfrak{L}^{\mathfrak{F}}(\mathscr{V}),\boldsymbol{\mathscr{F}}) \otimes \mathbb{I}_{m,\ell} \simeq \nu^* \mathrm{Gr}^{(\omega)}_{\beta_1 z^\omega}(L(\mathscr{V},\infty),\boldsymbol{\mathscr{F}})$ **as $2m\pi\mathbb{Z}$-equivariant local systems with Stokes structure.**

$\beta_1 z^\omega$, ν_1^* and ν_1 are computed explicitly (not unique).

$\upsilon < 1$

The stationary phase formula implies $0 \in D$.
Let $\mathscr{I}_0$ be the index set of the HLT decomposition of $\mathscr{V}$ at 0.
We obtain $(L(\mathscr{V},0),\mathscr{F}) \in \mathrm{Loc}^{\mathrm{St}}(\mathscr{I}_0)$ from $\mathscr{V}_{|U_0} \in \mathrm{MF}(U_0,0)$.

Theorem

There exist $\beta_1 z^{\omega} \neq 0$ $(\omega = \upsilon(\upsilon-1)^{-1} < 0)$, a bijection

$$\nu^* : \mathscr{I}_0(\beta_1 z^{\omega}) = \left\{ \beta_1 z^{\omega} + \sum_{0<a<|\omega|} \gamma_a z^{-a} \in \mathscr{I}_0 \right\} \simeq \mathscr{I}_\infty^{\mathfrak{F}}(\beta w^{\upsilon}),$$

and an affine isomorphism $\nu : \mathbb{R} \simeq \mathbb{R}$ such that

- $\nu^* : \left(\mathscr{I}_0(\beta_1 z^{\omega}), \leq_{\nu(\theta)}\right) \simeq \left(\mathscr{I}_\infty^{\mathfrak{F}}(\beta w^{\upsilon}), \leq_{\theta}\right)$ **for any $\theta \in \mathbb{R}$,**
- $\mathrm{Gr}^{(\upsilon)}_{\beta w^{\upsilon}}(\mathfrak{L}^{\mathfrak{F}}(\mathscr{V}),\infty) \otimes \mathbb{I}_{m,\ell} \simeq \nu^* \mathrm{Gr}^{(\omega)}_{\beta_1 z^{\omega}}(L(\mathscr{V},0),\mathscr{F})$ **as $2m\pi\mathbb{Z}$-equivariant local systems with Stokes structure.**

$\beta_1 z^{\omega}$, ν_1^* and ν_1 are computed explicitly (not unique).