Tame fundamental groups of rigid spaces

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Étale fundamental groups of rigid-analytic spaces can be challenging to understand. For example, $\pi_1^{\text{ét}}(D_{\mathbb{C}_p})$ of the affinoid unit disc over $\mathbb{C}_p$ is not topologically finitely generated, as

$$\mathrm{H}^1_{\text{ét}}(D_{\mathbb{C}_p}, \mathbb{F}_p) = \mathrm{Hom}(\pi_1^{\text{ét}}(D_{\mathbb{C}_p}), \mathbb{F}_p)$$

is infinite. For a proper smooth rigid-analytic space X over $\mathbb{C}_p$, the cohomology groups $\mathrm{H}^*_{\text{ét}}(X, \mathbb{F}_p)$ are finite (as shown by Scholze [15]), and the étale fundamental group is likewise expected to be topologically finitely generated.[1] However, it seems that we currently lack tools to show this, unless X is the analytification of an algebraic variety.

As in the case of schemes [13], one can deal with such issues by considering the tame quotient of the fundamental group. For us, a rigid-analytic space over a non-archimedean field K is an adic space locally of finite type over $\mathrm{Spa}(K, K^+)$. Thus, the residue fields of points of X are equipped with valuations, and the natural definition that presents itself (considered in [7]) is the following: an étale morphism of adic spaces $f\colon Y \to X$ is **tame** if for every $y \in Y$, the finite separable extension of valued fields $k(y)/k(f(y))$ is tamely ramified (meaning that $[k(y)^{\mathrm{sh}} : k(f(y))^{\mathrm{sh}}]$ is prime to the residue characteristic exponent). For X connected, tame finite étale maps $Y \to X$ form a Galois category whose fundamental group $\pi^{\mathrm{t}}(X)$ is a quotient of $\pi_1^{\text{ét}}(X)$.

However, with this definition, the tame fundamental group $\pi_1^{\mathrm{t}}(D_{\mathbb{C}_p})$ is still infinite! Indeed, the coverings defined by

$$y^p - y = \lambda x \qquad (\lambda \in \mathbb{C}_p \text{ with } |\lambda| = 1)$$

are tame (even unramified) and yield an infinite number of maps $\pi_1^{\mathrm{t}}(D_{\mathbb{C}_p}) \to \mathbb{F}_p$. Intuitively, the tameness condition introduced above measures only the ramification along the special fiber of a formal model of X, while in the presented example the wild ramification happens at infinity of the special fiber. We correct this by introducing the following notion: an étale morphism of rigid-analytic spaces $f\colon Y \to X$ is **tame relative to** K if for every maximal point $y \in Y$ and every valuation subring $V \subseteq k(y)^+$ containing K^+, the extension of valued fields $k(y)/k(f(y))$ is tamely ramified with respect to V. Again, for X connected, we obtain a Galois category whose fundamental group $\pi^{\mathrm{t}}(X/K)$ is a quotient of $\pi_1^{\mathrm{t}}(X)$.

For the unit disc $D_{\mathbb{C}_p}$, such test pairs (y, V) consist of points of $D_{\mathbb{C}_p}$ (continuous valuations on $K\langle x\rangle$ which are ≤ 1 on $K^+\langle x\rangle$) and one additional point corresponding to a rank two continuous valuation which is unbounded on $K^+\langle x\rangle$. In fact, if X is quasi-compact and separated, then the test pairs $(y^\circ, k(y)^+)$ form the set of points of an adic space (not a rigid space in general) $\overline{X}$ containing X, the **universal compactification** of X/K defined by Huber [6, §5.1]. Alternatively,

[1] We learned of this question from Bogdan Zavyalov.

$\overline{X}$ can be described as the inverse limit of all compactifications of special fibers of all formal models of X [11]. Thus $\pi_1^{\mathrm{t}}(X/K) = \pi_1^{\mathrm{t}}(\overline{X})$, whenever $\overline{X}$ exists.

Our main result is the following.

Theorem 1. *Let X be a connected qcqs rigid space over a non-archimedean field K. Suppose that the tame Galois group $\pi_1^{\mathrm{t}}(K) = \mathrm{Aut}(K^{\mathrm{t}}/K)$ is topologically finitely generated. Then $\pi_1^{\mathrm{t}}(X/K)$ is topologically finitely generated.*

Similarly, if K is algebraically closed, we can show the Künneth formula

$$\pi_1^{\mathrm{t}}(X \times Y/K) \simeq \pi_1^{\mathrm{t}}(X/K) \times \pi_1^{\mathrm{t}}(Y/K),$$

and that if L is an algebraically closed non-archimedean field containing K, then $\pi_1^{\mathrm{t}}(X_L/L) \simeq \pi_1^{\mathrm{t}}(X/K)$. In light of [3, 14] it is an interesting question whether $\pi_1^{\mathrm{t}}(X/K)$ is topologically finitely presented. Using our methods, we can show that this is the case if X is smooth and admits a semistable model such that the strata of its special fiber admit normal crossings compactifications. In this situation, there is a "van Kampen formula" expressing $\pi_1^{\mathrm{t}}(X/K)$ in terms of the more classically studied tame fundamental groups of the strata.

The proof of Theorem 1 relies on

(1) desingularization techniques [9, 16] which allow us to reduce the finite generation question to the case where $K = \overline{K}$ and X is smooth, with a semistable formal model $\mathfrak{X}$ (treated as a log formal scheme over K^+),
(2) a "semistable Abhyankar's lemma," relating $\pi_1^{\mathrm{t}}(X)$ (not $\pi_1^{\mathrm{t}}(X/K)$!) to the Kummer étale fundamental group $\pi_1^{\text{ét}}(\mathfrak{X}_0)$ of the log special fiber,
(3) an additional argument showing that $\pi_1^{\mathrm{t}}(X/K)$ is isomorphic to the tame Kummer étale fundamental group $\pi_1^{\text{ét,t}}(\mathfrak{X}_0/k)$ (a notion we needed to introduce along the way, and which I will not explain here),
(4) and finally, proving that for a suitable class of log schemes over an algebraically closed field k, the tame Kummer étale fundamental group is topologically finitely generated (see Theorem 2 below).

Logarithmic geometry beyond fs. A major obstacle to this approach is that since $K = \overline{K}$, it is not discretely valued, and hence the log special fiber $\mathfrak{X}_0$ will not be an fs log scheme, precluding the application of most of logarithmic geometry. Recall that a log scheme is fs if it locally admits a chart by an fs (finitely generated and saturated) monoid. Here, the log structure on K^+ admits a chart by the monoid Γ_K^+, the positive part of the value group. This monoid is not finitely generated; however, it is valuative and divisible, which turns out to be quite helpful in this context.

In order to overcome the obstacle, we needed to develop the foundations of logarithmic geometry beyond fs log schemes, which is a project by itself. (Similar, though less comprehensive, approaches appear in some recent papers [1, 2, 12].) The basic notion is that of an sfp morphism. A map of saturated monoids $P \to Q$ is **sfp** (finitely presented up to saturation) if Q is the saturation of a finitely presented monoid over P, or equivalently, if

$$Q = (P \oplus_{P_0} Q_0)^{\mathrm{sat}}$$

for a map of fs monoids $P_0 \to Q_0$ (these are precisely the compact objects of the category of saturated monoids over P). A map of saturated log schemes $Y \to X$ is **locally sfp** if it is étale locally of the form

$$\mathrm{Spec}(Q \to B) \to \mathrm{Spec}(P \to A)$$

where $(P \to A) \to (Q \to B)$ is a map of saturated prelog rings such that $P \to Q$ is sfp and $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q] \to B$ is finitely presented as a map of rings; it is **sfp** if it is locally sfp and qcqs. Crucially, we show that if $(P \to A) = \varinjlim (P_\alpha \to A_\alpha)$ is a filtered colimit of saturated prelog rings, then the category of sfp log schemes over $\mathrm{Spec}(P \to A)$ is the colimit of the system of categories of sfp log schemes over $\mathrm{Spec}(P_\alpha \to A_\alpha)$. Since for any given $(P \to A)$ we can find such a system with P_α fs and A_α finitely generated over $\mathbb{Z}$, this allows us to extend many known results from fs log schemes to sfp maps (analogously to the elimination of noetherian hypotheses in [5, §8]). Using this, we define smooth, étale, and Kummer étale maps, and develop the theory of the Kummer étale site and the Kummer étale fundamental group (see [8]) for arbitrary saturated log schemes.

Interestingly, sfp or Kummer étale maps might not be locally of finite type as maps of schemes. Indeed, there exist tame extensions of valued fields L/K such that Γ_L^+ is not finitely generated as a monoid over Γ_K^+, and the valuation ring L^+ is not finitely generated over K^+. (For example, let K be a non-archimedean field with $|2| = 1$, with value group $\mathbb{Z} + \mathbb{Z}\sqrt{2} \subseteq \mathbb{R}$, and let $L = K(\sqrt{x}, \sqrt{y})$ where $\nu(x) = 1$ and $\nu(y) = \sqrt{2}$.) However, the map $\Gamma_K^+ \to \Gamma_L^+$ is sfp, and with the natural log structures, the map $\mathrm{Spec}(L^+) \to \mathrm{Spec}(K^+)$ is Kummer étale. Surprisingly, thanks to fundamental results of Kato [10] and Tsuji [17], these difficulties go away for sfp log schemes over a base with a chart given by a divisible valuative monoid, such as $\mathrm{Spec}(K^+)$ for an algebraically closed non-archimedean field K.

Back to tame fundamental groups. With all these preparations, we can finish the proof of Theorem 1 by proving:

Theorem 2. *Let X be a connected log scheme which is sfp over* $\mathrm{Spec}(P \to k)$ *where P is a divisible valuative monoid with finitely many faces and k is an algebraically closed field. Then the tame Kummer étale fundamental group $\pi_1^{\text{ét,t}}(X/k)$ is topologically finitely generated.*

The proof uses "formal gluing" along the log stratification of X to reduce to the case of locally constant log structure, which in turn can be reduced to the case of trivial log structure. In this case, we use alterations to reduce to a result of Esnault and Kindler [4].

As is well-known, the proofs of all known results of (topological) finite generation of fundamental groups in algebraic geometry eventually rely on the finite generation of the topological fundamental group of smooth curves over $\mathbb{C}$. Ultimately, so does our proof, after a very long pipeline of reductions.

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