JAGS Fall 2019 exercises

Basics of étale and ℓ-adic cohomology

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Note: Italicized text indicates a remark. The b^{th} exercise for lecture a will often times be denoted 'exercise $a.b$'.

1. Introduction.

(1) Compute the number of points $|X(\mathbf{F}_q)|$ of $X = \mathrm{Gr}(k, n)$ (Grassmannian of k-dimensional linear subspaces in n-dimensional affine space) over the finite field $\mathbf{F}_q$ and the corresponding Hasse–Weil zeta function $Z(X, t)$.

Compare with singular cohomology of the complex Grassmannian if you already know how to compute it.

(2) Let X be a variety over $\mathbf{F}_q$. Show that

$$Z(X,t) = \prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}}.$$

Here the product runs over all closed points x of X, and $\deg(x)$ is the degree of the field extension $[\kappa(x) : \mathbf{F}_q]$.

(3) Show that if $Y \subseteq X$ is a closed subvariety and $U = X \setminus Y$ is the complementary open, then

$$Z(X,t) = Z(Y,t) \cdot Z(U,t).$$

(4) Let F be an endomorphism of a vector space V over some field K. Show the following equality of power series in $K[[t]]$:

$$\exp\left(\sum_{n \geq 1} \mathrm{Tr}(F^n)\frac{t^n}{n}\right) = \frac{1}{\det(1 - tF)}.$$

(5) Let $X \subseteq \mathbf{P}^N_{\mathbf{F}_q}$ be a projective variety over the finite field $\mathbf{F}_q$ and let $\overline{X} \subseteq \mathbf{P}^N_{\overline{\mathbf{F}}_q}$ be its base change to $\overline{\mathbf{F}}_q$. Let $F\colon \mathbf{P}^N_{\overline{\mathbf{F}}_q} \longrightarrow \mathbf{P}^N_{\overline{\mathbf{F}}_q}$ be the map defined in homogeneous coordinates by

$$F(x_0 \colon \ldots : x_N) = (x_0^q \colon \ldots : x_N^q).$$

Show that F maps $\overline{X}$ to itself and that its fixed points on $\overline{X}$ are precisely $X(\mathbf{F}_q)$.

(6) Compute the Hasse–Weil zeta function of the circle $X = \{x^2 + y^2 = 1\} \subseteq \mathbf{A}^2_{\mathbf{F}_p}$ and compare it with the zeta function of $X' = \mathbf{G}_{m,\mathbf{F}_p}$. Note that X and X' become isomorphic over $\overline{\mathbf{F}}_p$.

2. Morphisms.

(1) *For a scheme X an open subscheme $U \subseteq X$ has the property that its geometry is entirely determined by that of X and its underlying topological space $|U| \subseteq |X|$. This feature seems so important for open subschemes that one might imagine it characterizes them. This exercise shows this to be the case.*

Let U and X be Noetherian schemes and let $j : U \longrightarrow X$ be a morphism. Suppose that U represents the functor

$$S \mapsto \{f : S \longrightarrow X : f(|S|) \subseteq j(|U|)\}$$

(where $|\cdot|$ represents the underlying topological space). Use Grothendieck's characterization of open embeddings (open embeddings are precisely étale monomorphisms) to show that j is an open embedding.

This observation is also important for defining open subobjects in categories other than schemes (e.g. adic/rigid spaces).

(2) Let S be an integral normal scheme and let $f : X \longrightarrow S$ be an étale morphism with X connected. Use [Stacks, Tag025P] to show that X is integral. Give an example that shows this can fail if S is not assumed to be normal (Hint: consider the nodal cubic curve $V(y^2 - x^2(x+1)) \subseteq \mathbf{A}^2_{\mathbf{C}}$ and try drawing a picture for some of its 'covers').

(3) Let $f : X \longrightarrow Y$ be a morphism of varieties over an algebraically closed field k. Let $x \in X(k)$ and let $y := f(x)$. If f is étale at x show that the induced map $\widehat{f_x} : \widehat{\mathcal{O}}_{Y,f(x)} \longrightarrow \widehat{\mathcal{O}}_{X,x}$ is an isomorphism. Does this remain true if one does not assume that k is algebraically closed? Does it remain true if X and Y are arbitrary schemes?

(4) Let k be a field of arbitrary characteristic and let E be an elliptic curve over k. Show that the multiplication-by-n map $[n] : E \longrightarrow E$ is étale if and only if $\mathrm{char}(k) \nmid n$.

(5) Let k be a field of arbitrary characteristic and let $\mathbf{G}_{m,k}$ denote the usual multiplicative group over k. For what n is the multiplication-by-n map $[n] : \mathbf{G}_{m,k} \longrightarrow \mathbf{G}_{m,k}$ étale?

(6) Let $\mathrm{SL}_{n,\mathbf{Z}}$ be the functor which associates to a ring R the set $\mathrm{SL}_n(R)$ of $n \times n$-matrices over R with determinant 1. Show that $\mathrm{SL}_{n,\mathbf{Z}}$ is representable and use the infinitesimal lifting criterion to show that $\mathrm{SL}_{n,\mathbf{Z}} \longrightarrow \mathrm{Spec}(\mathbf{Z})$ is smooth.

(7) Use Example 4.10 of [Mil] to show that the henselization of $\mathbb{Z}_{(p)}$ is the integral closure of $\mathbb{Z}_{(p)}$ in $\mathbb{Z}_p$. What is the henselization of $k[t]_{(t)}$?

(8) Use the infinitesimal lifting criterion to prove the following version of Hensel's lemma: let $(A, \mathfrak{m})$ be a complete local ring and let $X \longrightarrow \mathrm{Spec}(A)$ be smooth (resp. étale)). Show that the map $X(A) \longrightarrow X(A/\mathfrak{m})$ is surjective (resp. bijective). Use this to show the normal version of Hensel's lemma: let $f(t) \in \mathbb{Z}_p[t]$ be a polynomial such that $\overline{f(t)} \in \mathbb{F}_p[t]$ is separable, then every root of $\overline{f(t)}$ has a unique lift to $\mathbb{Z}_p$.

(9) Let X be a smooth variety over k an arbitrary field. Show that $X(k^{\mathrm{sep}})$ is dense in X (Hint: show that this condition is insensitive to étale morphisms and reduce to $\mathbf{A}_k^{\dim(X)}$).

(10) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms. Suppose that $g \circ f$ and g are étale. Show that f is étale.

(11) Let S be any scheme over $\mathbb{F}_p$. Denote by F_S the *absolute Frobenius map* $S \longrightarrow S$ which is the identity on the underlying topological space and is the p^{th}-power map on $\mathcal{O}_S$ (i.e. on an affine open $\mathrm{Spec}(R) \subseteq S$ it is the map induced by the map $R \longrightarrow R$ given by $r \mapsto r^p$). We say that S is *perfect* if F_S is an isomorphism.

Let $f : X \longrightarrow S$ be an S-scheme. We then define the scheme $X^{(p)}$ by the following cartesian diagram

$$\begin{array}{ccc} X^{(p)} & \longrightarrow & X \\ \downarrow & & \downarrow f \\ S & \xrightarrow{F_S} & S \end{array}$$

We then obtain a *relative Frobenius map* $F_{X/S} : X \longrightarrow X^{(p)}$ by declaring its projection to X be F_X and its projection to S be f. We would like to understand $F_{X/S}$ when S is perfect (which we assume in the following).

a) Compute $X^{(p)}$ and $F_{X/S}$ when $X = \mathbb{A}^d_S$ for some $d \geqslant 0$.

b) Use exercise 10. and Grothendieck's characterization of open embeddings to show that if $f : X \longrightarrow S$ is étale then $F_{X/S}$ is an isomorphism.

c) Combine a) and b) to show that if f is smooth of relative dimension d then $F_{X/S}$ is locally free of rank p^d.

d) Show that if X and Y are varieties over an algebraically closed field k and $f : X \longrightarrow Y$ is an étale morphism then for all $x \in X(k)$ the induced map $df_x : T_x X \longrightarrow T_{f(y)} Y$ is an isomorphism.Is this true if k is not algebraically closed? Is this true for non-closed points? Is this true if X and Y are arbitrary schemes.

(12) Let X and Y be schemes and let $f : X \longrightarrow Y$ be an étale morphism. Let us say that f is *locally factorizable* if for all $x \in X$ there exists a neighborhood U of x and V of $f(x)$ such that $f(U) \subseteq V$ and for which there is a factorization

$$\begin{array}{ccccc} & & W & & \\ & \overset{j}{\nearrow} & & \overset{\pi}{\searrow} & \\ U & & \xrightarrow{\quad f \quad} & & V \end{array}$$

with j an open embedding and π a finite étale map. Is every étale morphism locally factorizable?

In other words, the question is whether the étale site of a scheme X has a subcategory $\mathscr{C}$ of simpler objects (open embeddings followed by finite étale) which is sufficient for defning sheaves (in the sense that every object of X_{et} has a refinement by an object of $\mathscr{C}$).

(13) Show that if $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ are local rings and $f : (A, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$ is flat, then it's injective and, in particular, faithfully flat.

(14) Let $f : X \longrightarrow Y$ be faithfully flat. Show that f is an epimorphism in the category of schemes (Hint: use the exercise 2.13).

3. Étale fundamental group (I).

(1) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be such that $g \circ f$ and g are finite étale. Prove that f is finite étale.

(2) Let S be a scheme and let $\bar{s}$ be a geometric point of S. Show that if $f_i : X_i \longrightarrow S$ are finite étale covers and $u : X_1 \longrightarrow X_2$ is an S-morphism such that the induced map $u_* : F_{\bar{s}}(X_1) \longrightarrow F_{\bar{s}}(X_2)$ is an isomorphism, then u is an isomorphism.

(3) Let S be a connected scheme and let $X \longrightarrow S$ be a connected finite étale cover. Show that any S-morphism $u : X \longrightarrow X$ is an isomorphism.

(4) Show that if $(S_1, \overline{s_1}) \longrightarrow (S_2, \overline{y_2})$ is a mapping of connected pointed schemes, then the induced map $\pi_1^{\acute{e}t}(S_1, \overline{s_1}) \longrightarrow \pi_1^{\acute{e}t}(S_2, \overline{s_2})$ is surjective if and only if for all connected finite étale covers $X_2 \longrightarrow S_2$ the space $S_1 \times_{S_2} X_2$ is connected. Find an aanlogue of when a map on fundamental groups is injective.

(5) Use exercise 3.4, together with exercise 2.2 to show that if $(S, \bar{s})$ is a pointed scheme with S integral and normal and if $U \subseteq S$ is open, then the induced map $\pi_1^{\acute{e}t}(U, \bar{s}) \longrightarrow \pi_1^{\acute{e}t}(S, \bar{s})$ is surjective. Is this true if one doesn't assume that S is normal?

(6) Classify explicitly the étale covers of $\operatorname{Spec}(\mathbf{Z}_p)$ and use this to compute $\pi_1^{\acute{e}t}(\operatorname{Spec}(\mathbf{Z}_p))$.

(7) Can you give a description of all the finite étale covers of $\operatorname{Spec}(\mathbf{Z}[\frac{1}{n}])$ and describe $\pi_1^{\acute{e}t}(\operatorname{Spec}(\mathbf{Z}[\frac{1}{n}]))$ in terms of $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$? In particular, can you describe $\pi_1^{\acute{e}t}(\operatorname{Spec}(\mathbf{Z}))$ (Hint: the Minkowski bound).

(8) Can you give a description of $\pi_1^{\acute{e}t}(\operatorname{Spec}(\mathbf{C}(\!(t)\!)))$? Can you interpret this geometrically?

(9) Let π be a profinite group. Is the natural map $\pi \longrightarrow \widehat{\pi}$ an isomorphism? Can you give an example?

(10) Let p be a prime and let K be a field of characteristic different than p. Let $K(\mu_{p^\infty})$ be the Galois extension of K given by adjoining the roots of $x^{p^n} - 1$ for all $n \geqslant 1$. Show that there is a natural embedding $\chi_p : \operatorname{Gal}(K(\mu_{p^\infty})/K) \longrightarrow \mathbf{Z}_p^\times$ called the *cyclotomic character*. Is it true that χ_p either has finite image or cofinite image?

4. Étale fundamental group (II).

(1) Compute $\pi_1^{\acute{e}t}(\operatorname{Spec}(O_{C,x}))$ where C is a smooth projective curve over $\mathbf{C}$ and $x \in C(\mathbf{C})$. Your answer shoudl involve the projective geometry of C itself.

(2) Try and compute $\pi_1^{\acute{e}t}(\mathbf{A}^2_{\mathbf{C}} - \{0\})$ (this is difficult without some big machinery).

(3) Try and compute $X = \pi_1^{\acute{e}t}(\mathbf{Z}_p[x, y]/(xy - p))$ and understand the relationship between $\pi_1^{\acute{e}t}(X_{\mathbf{Q}_p})$ and $\pi_1^{\acute{e}t}(X_{\mathbf{F}_p})$ (note that X is not proper so that Grothendieck's specialization results don't apply).

(4) It is a somewhat hard fact that if X is a proper connected variety over an algebraically closed field k and k'/k is an algebraically closed extension, then the natural map $\pi_1^{\acute{e}t}(X_{k'}) \longrightarrow \pi_1^{\acute{e}t}(X)$ is an isomorphism (e.g. see [Tag0A49,Stacks]—the proof uses ideas that come up in the proof of Grothendiecks' specialization theorem). Show that this result is false for $X = \mathbf{A}^1_{\overline{\mathbf{F}_p}}$ and $k' = \overline{\mathbf{F}_p(T)}$.

(5) Compute $\pi_1(C)^{(p)}$ where C is a smooth proper integral curve over an algebraically closed field k, p is the characteristic of k (which can be zero), and for $p > 0$ $\pi_1^{\acute{e}t}(C)^{(p)}$ means the maximal pro-prime-to-p quotient of $\pi_1^{\acute{e}t}(C)$ (by definition $\pi_1^{\acute{e}t}(C)^{(0)} = \pi_1^{\acute{e}t}(C)$). Note that your proof will use the Riemann existence theorem, the classical computation of the fundamental groups of real surfaces, Grothendieck's specialization results, and the result [Tag0A49,Stacks] cited in exercise 4.4.

(6) If X is a normal integral scheme and U is an open subset of X, then the map $\pi_1^{\acute{e}t}(U) \longrightarrow \pi_1^{\acute{e}t}(X)$ is surjective (e.g. see [Tag0BQI,Stacks]. Give a counterexample to this claim when X is not normal.

(7) Let $f \in k[x_0, \ldots, x_n]$ be a homogenous polynomial. Let $\widetilde{X} := V(f) \subseteq \mathbf{A}^{n+1}_k$ and $X := V(f) \subseteq \mathbf{P}^n_K$. Show that the natural map $\widetilde{X} \longrightarrow X$ induces a surjection $\pi_1^{\acute{e}t}(\widetilde{X}) \longrightarrow \pi_1^{\acute{e}t}(X)$. Can you describe the kernel of this map?

5. Étale topology (I).

6. Étale topology (II).

7. Cohomology of curves and abelian varieties.

8. Artin comparison and Artin vanishing.

9. Proper base change and nearby cycles.

10. Properties of ℓ-adic cohomology.

11. Cohomology over finite fields. Lefschetz trace formula and the Weil conjectures.

12. Purity and the Riemann hypothesis.

Prerequisites

Schemes and sheaf cohomology, basics of homological algebra.

Bibliography

[Stacks] *The stacks project*, https://stacks.math.columbia.edu

[Mil] *Étale cohomology*, J. Milne