Symplectic geometry and symplectic reduction - notes

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1 Introduction

These notes about symplectic geometry and symplectic redutions has been written for the Junior Algebraic Geometry Seminar. Throughout these notes we will be assuming the following.

G is a connected reductive complex linear algebraic group

- A *complex linear algebraic group over* is a subgroup of $GL_n(\mathbb{C})$, defined by a set of polynomial equations.
- Reductive means all (complex) representations split as sums of irreducible ones.
- Assumptions on G imply that G is a complexification of a compact real subgroup $K < G$, $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$.

In the example we consider we will always take $K < G$ to be:

$$S^1 < \mathbb{C}^*, (S^1)^m < (\mathbb{C}^*)^m, SU(m) < SL(m, \mathbb{C})$$

We consider the action of G on a projective variety X and assume that G acts via SL-transformations on the projective space, i.e.

$$\begin{array}{ccc} G & \curvearrowright & X \\ \downarrow & & \downarrow \\ SL(n+1,\mathbb{C}) & \curvearrowright & \mathbb{P}^n \end{array}$$

The main reference for the symplectic geometry part are [H] and [MD-S] and for the algebraic geometry/GIT part [T]. Some more useful references are listed at the end.

2 Notation

- X is a smooth projective variety
- M is a smooth symplectic manifold
- G is a complex connected reductive linear algebraic group
- K is a compact connected Lie group whose complexification is G
- $T \subseteq K$ is a maximal torus in K
- ω denotes the symplectic form
- μ_K is the moment map for the action of K
- $X /\!\!/ K = \mu_K^{-1}(0)/K$ denotes the symplectic reduction

3 Why symplectic reduction? / Quotients in algebraic geometry

Suppose we want to define a quotient X/G. The naive thing to do is to take the orbit space - it gives us a well-defined quotient in the category of topological spaces (the topology is defined via the quotient map $X \to X/G$). But since we started with X being a projective variety and chose a very good group action, we would like our quotient to be defined in the category of projective varieties. The orbit space is obviously not good, because the orbit space is almost always non-Hausdorff, this happens whenever some orbit has a smaller-dimensional orbit in its closure.

Question 1. *Can we fix this by simply removing the smaller dimensional orbits which sit in closures?*

Example 1. Let $\mathbb{C}^* \curvearrowright \mathbb{P}^2$ by $SL(2,\mathbb{C})$ transformations via:

$$\mathbb{C}^* \ni \lambda \mapsto \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

The orbits are: $\{0\}$, $\{x = 0\} \setminus \{0\}$,$\{y = 0\} \setminus \{0\}$, $\{xy = \alpha\}_{\alpha \in C^*}$. We obviously want the quotient to be $\mathbb{C}$, but we have a triple point at 0. Even if we remove the smaller-simensional orbit $\{0\}$, we still get a double point in the orbit space.

There are several standard solutions to the problem of defining a quotient for the actions of reductive groups on projective varieties:

- Hilbert quotients and Chow quotients
- GIT quotient
- Symplectic reduction

Under reasonable assumptions the GIT quotient and the symplectic reduction are isomorphic.

4 Preliminaries on symplectic manifolds

Definition 1 (Symplectic manifold)**.** A smooth manifold M is called *symplectic* if it is equipped with a *symplectic form* ω, i.e. a closed, nondegenerate, skew-symmetric differential 2-form. Morphisms in the category of symplectic manifolds are *symplectomorphisms*:

$$\psi \in Symp(M_1, M_2) \iff \psi^*\omega_2 = \omega_1.$$

Let K be a Lie group with Lie algebra $\mathfrak{k}$ and denote by $\mathfrak{k}^*$ the dual of $\mathfrak{k}$ with respect to the natural pairing $\langle ., . \rangle : \mathfrak{k}^* \times \mathfrak{k} \to \mathbb{R}$.

Definition 2 (Hamiltonian action)**.** Assume K acts on M by symplectomorphisms. The action is called *Hamiltonian* if there exists a *moment map* $\mu : M \to \mathfrak{k}^*$ satisfying:

1. μ is K-equivariant (with respect to the coadjoint action on $\mathfrak{k}^*$)
2. $$dH_\xi = \iota_{X_\xi}\omega,$$
   where $H_\xi : M \to \mathbb{R}$ is defined as $H_\xi(m) = \langle \mu(m), \xi \rangle$ for $\xi \in \mathfrak{k}$. The right-hand side is the contraction of ω with the vector field X_ξ associated to $\xi \in \mathfrak{k}$.

Remark 1 (historical)**.** In less recent literature (or the more physically inclined one) the vector field X_ξ is sometimes called the *infinitesimal action.*

Example 2 (The most classical Example)**.** Consider the action of $SO(3)$ on $\mathbb{R}^3$ by rotations and extend this action to an action on the cotangent bundle

$T^*\mathbb{R}^3 = \mathbb{R}^6$ (with coordinates $\{q_1, q_2, q_3, p_1, p_2, p_3\}$). Then the vector field generated by the action is

$$X_{L_i} = \sum_{j,k} \epsilon_{ijk} \Big(p_j \frac{\partial}{\partial p_k} + q_j \frac{\partial}{\partial p_k} \Big),$$

and one can easily check that the map $\mu : \mathbb{R}^6 \to \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ given by

$$\mu(\mathfrak{q}, \mathfrak{p}) = \mathfrak{q} \times \mathfrak{p}$$

is the moment map for this action.

Example 3 (Torus action on $\mathbb{C}^n$ and $\mathbb{P}^n$)**.** Consider the action of $(S^1)^m$ on $\mathbb{C}^n$ given by

$$(t_1, \dots, t_n)(z_1, \dots, z_n) = (t_1^{k_1} z_1, \dots, t_n^{k_n} z_n), k_i \in \mathbb{Z}.$$

The moment map for this action is

$$\mu(\mathbf{z}) = \frac{1}{2}(k_1|z_1|^2, \dots, k_n|z_n|^2)$$

Analogously for the action on $\mathbb{P}^n$

$$(t_1, \dots, t_n)[z_1 : \dots : z_n] = [t_1^{k_1} z_1 : \dots : t_n^{k_n} z_n],$$

the moment map is

$$\mu(\mathbf{z}) = \frac{1}{2|z|^2}(k_1|z_1|^2, \dots, k_n|z_n|^2)$$

Example 4 (Action of $U(n)$ on $\mathbb{C}^n$)**.** Consider the standard action of $U(n)$ on $\mathbb{C}^n$. The moment map for this action is

$$\mu(\mathbf{z})(A) = \frac{i}{2}\mathbf{z}^* A \mathbf{z}$$

Example 5 (Action of $U(n)$ on $\mathbb{P}^{n-1}$)**.** Consider the standard action of $U(n)$ on $\mathbb{C}^n$ and consider the induced action on $\mathbb{P}^{n-1}$. The moment map for this action is

$$\mu(\mathbf{z})(A) = \frac{i}{2|z|^2}\mathbf{z}^* A \mathbf{z}$$

The two examples above give us recipies to compute the moment maps for any linear action of on a subset on $\mathbb{C}^n$ or $\mathbb{P}^n$, by restriction and projection of the moment maps above: If $X \subseteq \mathbb{P}^n$, K acts on X by $U(n)$ transformations on $\mathbb{P}^{n-1}$, then if $\mu_U : \mathbb{P}^{n-1} \to \mathfrak{u}(n)$ is the moment map for $U(n)$, then the moment map for K acting on X is given by

$$X \xrightarrow{\mu_{U|X}} \mathfrak{u}(n)^* \to \mathfrak{k}^*,$$

where the last arrow is the canonical projection.

4.1 Elementary properties of moment maps

1. Let $H \to K$ be a Lie group homomorphism and $p_* : \mathfrak{k}^* \to \mathfrak{h}^*$ the induced Lie algebra homomorphism. Then if K acts on M with moment map μ_K, then the induced action of H is also Hamiltonian, with moment map $\mu_H = p^* \circ \mu_K$.

2. Let M_1, M_2 be two symplectic manifolds equipped with Hamiltonian actions of K with moment maps $\mu_1 : M_1 \to \mathfrak{k}^*$ and $\mu_2 : M_2 \to \mathfrak{k}^*$. Then the diagonal action of K on $M_1 \times M_2$ is Hamiltonian with moment map $\mu_1 + \mu_2$.

3. If K, H act on M with moment maps μ_K, μ_H and the actions commute, then $K \times H$ acts on M with moment map $\mu_K \oplus \mu_H : M \to \mathfrak{k}^* \oplus \mathfrak{h}$.

Theorem 1 (Atiyah-Guillemin-Sternberg). *Let $T = (S^1)^m$ act on a compact connected symplectic manifold M in a Hamiltonian way. Then the image of the moment map is a convex polyhedron (the covex hull of the images of the fixed points of the action).*

5 Symplectic reduction

Definition 3 (Symplectic reduction). Let M be a compact symplectic manifold equipped with a Hamiltonian action of a compact Lie group K, with moment map $\mu : M \to \mathfrak{k}^*$. Assume 0 is a regular value of μ.

$$M /\!\!/ K := \mu^{-1}(0)/K.$$

The assumption that 0 is a regular value of the moment map is equivalent to each $m \in \mu^{-1}(0)$ having a finite stabilizer. It ensures that $\mu^{-1}(0)$ is a manifold and G acts locally freely on it. In particular, the symplectic reduction $M /\!\!/ K$ is (at least) an orbifold, and morover it's a symplectic orbifold - there is an induced symplectic structure in $M /\!\!/ K$.

Example 6 ($\mathbb{P}^n$ as symplectic reduction of $\mathbb{C}^{n+1}$). Let $U(1) < \mathbb{C}^*$ act on $\mathbb{C}^{n+1}$ the standard way. The moment map is $\mu(z) = |z|^2 + c$ (for any $c \in \mathbb{R}$). Choose $a = -1$, then

$$\mathbb{C}^{n+1} /\!\!/ U(1) = \frac{|z|^2 = 1}{U(1)} = \frac{S^{2n+1}}{U(1)} = \mathbb{P}^n$$

Example 7 ($Gr(k,n)$)**.** Let $U(k)$ act on $Hom(\mathbb{C}^k, \mathbb{C}^n)$ (by matric multiplication). The moment map $\mu : Hom(\mathbb{C}^k, \mathbb{C}^n) \to \mathfrak{u}(k)^* \simeq \mathfrak{u}(k)$ is given by

$$\mu(A) = i(AA^* - Id).$$

The elements of $\mu^{-1}(0)$ are unitary k-tuples of vectors in $\mathbb{C}^n$, diviing by the action of $U(k)$ gives us

$$Hom(\mathbb{C}^k, \mathbb{C}^n)/\!\!/U(k) = Gr(k,n)$$

There is an important map, called the *Kirwan map*, associated to the symplectic reduction, relating the cohomology of the symplectic reduction with the equivariant cohomology of the original manifold.

Definition 4 (Kirwan map)**.** The Kirwan map

$$\kappa : H_K^*(M) \to H^*(\mu^{-1}(0)/K),$$

is defined as the composition $(\pi^*)^{-1} \circ i^*$,

$$\kappa : H_K^*(M) \xrightarrow{(i^*)} H_K^*(\mu^{-1}(0)) \xrightarrow{(\pi^*)^{-1}} H^*(\mu^{-1}(0)/K),$$

where i^* is the map of equivariant cohomology induced by the inclusion $i : \mu^{-1}(0) \to M$ and π^* is the natural isomorphism $H_K^*(\mu^{-1}(0)) \to H^*(\mu^{-1}(0)/K)$ induced by the quotient map $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/K$.

6 Symplectic reduction vs GIT quotient

Assume, as before, that we are in the following situation:

$$\begin{array}{ccccc} K & < & G & \curvearrowright & X \\ & & \downarrow & & \downarrow \\ SU(n+1) & < & SL(n+1,\mathbb{C}) & \curvearrowright & \mathbb{P}^n \end{array}$$

where $K < G$ is a compact real subgroup whose complexification is G, X a projective variety.

In this situation the action of K in $\mathbb{P}^n$ preserves the standard symplectic form ω (Fubini-Study) and moreover K acts by symplectomorphisms on X.

Theorem 2 (Kempf-Ness)**.** *Under all the assumptions above we have:*

1. *A G-orbit contains a zero of the moment map iff. it is polystable. It is unique up to the action of K.*

2. *A G-orbit is semistable iff. its closure contains a zero of the moment map, this zero is in the unique polystable orbit orbit in the closure of the original orbit.*

In particular, the GIT quotient $X/\!\!/G$ and the symplectic reduction $X/\!\!/K$ are equal (as sets).

References

[H] G. Heckman *Symplectic Geometry*, www.math.ru.nl/ heckman/symplgeom.pdf

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[K] F. Kirwan *Cohomology of Quotients in Symplectic and Algebraic Geometry*

[T] R. Thomas *Notes on GIT and symplectic reduction for bundles and varieties*, arXiv:math/0512411