Problem Set 4 due November 18, 2020

Affinoid algebras

Problem 1. Let A be an affinoid K-algebra and let $a \in A$. Show that $|a|_{\sup} < 1$ if and only if $\lim a^n = 0$. (The latter means that $\lim |a^n|_{\alpha} = 0$ for some/every residue norm $|\cdot|_{\alpha}$.)

Sites

Problem 2. Construct an equivalence of categories $Sh(\mathbf{R}) \simeq Sh^{adm}(\mathbf{Q})$.

Problem 3 (Sheaves on a base, site version). Let \mathscr{C} be a site and let $\mathscr{C}_0 \subseteq \mathscr{C}$ be a full subcategory closed under fiber products. Suppose that every object $c \in \text{ob } \mathscr{C}$ admits a covering family $\{c_{\alpha} \to c\}_{\alpha \in I}$ with every $c_{\alpha} \in \text{ob } \mathscr{C}_0$. Endow \mathscr{C}_0 with the induced topology: a family $\{c_{\alpha} \to c\}$ is a covering family if it is a covering family in \mathscr{C} . Prove that the inclusion functor induces an equivalence $\operatorname{Sh}(\mathscr{C}) \simeq \operatorname{Sh}(\mathscr{C}_0)$.

Blowing up

For the following exercises, it will be helpful to brush up on blow-ups, e.g. [Hartshorne, Chapter II 7, pp. 160–169] and on the valuative criteria of separatedness and properness [Chapter II 4].

Problem 4. Let X be a Noetherian scheme.

- (a) Let Y, Z ⊆ X be closed subschemes and let X' = Bl_{Y∩Z} X be the blow-up of their intersection. Prove that the strict transforms Ỹ, Ž̃ of Y and Z in X' are disjoint.
- (b) Suppose that X is integral, and let f ∈ K(X) be a nonzero rational function on X. Prove that there exists a blow-up X' = Bl_W → X which admits an open cover X' = X'₊ ∪ X'₋ such that f is a regular function on X'₊ and f⁻¹ is a regular function on X'₋.

Problem 5 (Riemann–Zariski space). Let X be a separated integral scheme of finite type over a field k, and let K be the field of rational functions on X.

- (a) Show that nontrivial blow-ups X' → X of X form a cofiltering subcategory 𝔅_X of the slice category Sch_{/X}.
- (b) Consider the topological space (called the Riemann-Zariski space)

$$\mathbf{ZR}(X) = \lim_{X' \to X \in \mathscr{B}_X} |X'| \in \mathbf{Top}.$$

Hint: Use the characterization of *a* such that $|a|_{sup} \leq 1$ proved in the lecture. Try to reduce to this by rescaling.

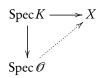
Hint: Compare with [Vakil, Theorem 2.5.1]. Its proof uses stalks at points, which you cannot do here, so you need a different argument.

If you really get stuck, try [SGA4 Vol. I Exposé III, Théorème 4.1]. You may also find the Stacks Project helpful.

= [Hartshorne, Exercise II 7.12]

Hint: In part (b), use Problem 4 to construct O, and the valuative criterion of properness to construct a point of ZR(X).

cofiltering = Every two objects are dominated by a third one, and every two parallel arrows can be equalized. Construct a bijection between points of $\mathbf{ZR}(X)$ and the set of valuation subrings $\mathcal{O} \subseteq K$ such that there exists a dotted arrow making the triangle below commute



(the dotted arrow is unique if it exists, thanks to the valuative criterion of properness). We say that the valuation subring $\mathcal{O} \subseteq K$ has center on X.

(c) Endow the set of valuation subrings $\mathcal{O} \subseteq K$ with center on X with the topology generated by the subsets

$$X(f) = \{ \mathcal{O} : f \in \mathcal{O} \} \quad \text{for} \quad f \in K.$$

Prove that the bijection constructed in (b) is a homeomorphism.