

Problem Set 1

due October 23, 2020

Problem 1. A *norm* on a field K is a map $|\cdot|: K \rightarrow [0, \infty)$ such that $|x| = 0$ iff $x = 0$, $|xy| = |x| \cdot |y|$, and $|x + y| \leq |x| + |y|$ for all $x, y \in K$. Show that the following are equivalent:

- (a) $|\cdot|$ is non-Archimedean, i.e. $|x + y| \leq \max\{|x|, |y|\}$ for $x, y \in K$,
- (b) $|n| \leq 1$ for all $n \in \mathbf{Z}$,
- (c) $\mathcal{O} = \{x \in K : |x| \leq 1\} \subset K$ is a subring of K .

Problem 2. Show that $\mathbf{Z}_p \simeq \mathbf{Z}[[X]] / (X - p)$.

Problem 3. Let $|\cdot|_1, |\cdot|_2: K \rightarrow [0, \infty)$ be two *nontrivial* non-Archimedean norms on a field K . Show that the following are equivalent:

- (a) $|\cdot|_1$ and $|\cdot|_2$ define the same topology on K ,
- (b) There exists a $c > 0$ such that $|\cdot|_1 = |\cdot|_2^c$,
- (c) $\{x \in K : |x|_1 \leq 1\} \subseteq \{x \in K : |x|_2 \leq 1\}$.

Problem 4. Describe the algebraic closure of $\mathbf{C}((t))$ and its completion in terms of Laurent series in fractional powers of t (Puiseux series).

Hint: Show that every finite extension of $\mathbf{C}((t))$ is of the form $\mathbf{C}((s))$ with $s^n = t$.

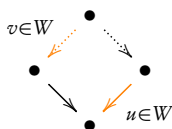
Problem 5 (Calculus of fractions). Let \mathcal{C} be a category and let $W \subseteq \mathcal{C}$ be a subcategory containing all isomorphisms and satisfying the two-out-of-three property: if f and g are composable arrows in \mathcal{C} and two of f , g , gf are in W then so is the third. There exists a functor

$$\mathcal{C} \rightarrow \mathcal{C}[W^{-1}]$$

which is initial among all functors $\mathcal{C} \rightarrow \mathcal{D}$ sending morphisms in W to isomorphisms; the category $\mathcal{C}[W^{-1}]$ is called the *localization* of \mathcal{C} in W .

We say that (\mathcal{C}, W) admits a *calculus of right fractions* if:

- a) every pair of solid arrows as below with $u \in W$ can be completed to a commutative square

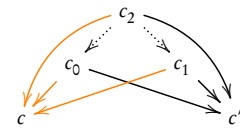


with $v \in W$, and

- b) for every pair of parallel morphisms $f, g: X \rightarrow Y$ in \mathcal{C} and every map $u: Y \rightarrow Z$ in W such that $uf = ug$ there exists a map $v: W \rightarrow X$ in W such that $fv = gv$.

Prove that if W admits a calculus of right fractions, then the localization $\mathcal{C}[W^{-1}]$ admits the following explicit description: the objects of $\mathcal{C}[W^{-1}]$ are the objects of \mathcal{C} , and the morphisms $c \rightarrow c'$ in $\mathcal{C}[W^{-1}]$ are equivalence classes of “roofs” $c \leftarrow c_0 \rightarrow c'$ with the backwards map in W , where $c \leftarrow c_0 \rightarrow c'$ and $c \leftarrow c_1 \rightarrow c'$ are equivalent if there exists a third $c \leftarrow c_2 \rightarrow c'$ and maps $c_0 \leftarrow c_2 \rightarrow c_1$ making the resulting diagram commute. Given $c \leftarrow c_0 \rightarrow c'$ and $c' \leftarrow c_1 \rightarrow c''$, applying axiom a) to $c_0 \rightarrow c' \leftarrow c_1$ gives $c_0 \leftarrow c_2 \rightarrow c_1$, and the composition is $c \leftarrow c_0 \leftarrow c_2 \rightarrow c_1 \rightarrow c''$.

Equivalence:



Composition:

