Problem Set 1

due October 23, 2020

Problem 1. A *norm* on a field K is a map $|\cdot|: K \to [0,\infty)$ such that $|x| = 0$ iff $x = 0$, $|xy| = |x|\cdot|y|$, and $|x+y| \leq |x| + |y|$ for all $x, y \in K$. Show that the following are equivalent:

(a) $|\cdot|$ is non-Archimedean, i.e. $|x+y| \leq \max\{|x|,|y|\}$ for $x, y \in K$,

(b) $|n| \leq 1$ for all $n \in \mathbf{Z}$,

(c) $\mathscr{O} = \{x \in K : |x| \leq 1\} \subset K$ is a subring of K.

Problem 2. Show that $\mathbf{Z}_p \simeq \mathbf{Z}[[X]]/(X-p)$.

Problem 3. Let $|\cdot|_1, |\cdot|_2 : K \to [0,\infty)$ be two *nontrivial* non-Archimedean norms on a field K. Show that the following are equivalent:

(a) $|\cdot|_1$ and $|\cdot|_2$ define the same topology on K,

(b) There exists a $c > 0$ such that $|\cdot|_1 = |\cdot|_2^c$,

(c) $\{x \in K : |x|_1 \leq 1\} \subseteq \{x \in K : |x|_2 \leq 1\}$.

Problem 4. Describe the algebraic closure of $\mathbf{C}((t))$ and its completion in terms of Laurent series in fractional powers of t (Puiseux series).

Hint: Show that every finite extension of $\mathbf{C}((t))$ is of the form $\mathbf{C}((s))$ with $s^n = t$.

Problem 5 (Calculus of fractions)**.** Let $\mathscr{C}$ be a category and let $W \subseteq \mathscr{C}$ be a subcategory containing all isomorphisms and satisfying the two-out-of-three property: if f and g are composable arrows in $\mathscr{C}$ and two of f, g, gf are in W then so is the third. There exists a functor

$$\mathscr{C} \to \mathscr{C}[W^{-1}]$$

which is initial among all functors $\mathscr{C} \to \mathscr{D}$ sending morphisms in W to isomorphisms; the category $\mathscr{C}[W^{-1}]$ is called the *localization* of $\mathscr{C}$ in W.

We say that $(\mathscr{C}, W)$ admits a *calculus of right fractions* if:

a) every pair of solid arrows as below with $u \in W$ can be completed to a commutative square

$$\begin{tikzcd}
& \bullet \arrow[dl, dotted, "v\in W"'] \arrow[dr, dotted] & \\
\bullet \arrow[dr] & & \bullet \arrow[dl, "u\in W"] \\
& \bullet &
\end{tikzcd}$$

with $v \in W$, and

b) for every pair of parallel morphisms $f, g : X \to Y$ in $\mathscr{C}$ and every map $u : Y \to Z$ in W such that $uf = ug$ there exists a map $v : W \to X$ in W such that $fv = gv$.

Prove that if W admits a calculus of right fractions, then the localization $\mathscr{C}[W^{-1}]$ admits the following explicit description: the objects of $\mathscr{C}[W^{-1}]$ are the objects of $\mathscr{C}$, and the morphisms $c \to c'$ in $\mathscr{C}[W^{-1}]$ are equivalence classes of "roofs" $c \leftarrow c_0 \to c'$ with the backwards map in W, where $c \leftarrow c_0 \to c'$ and $c \leftarrow c_1 \to c'$ are equivalent if there exists a third $c \leftarrow c_2 \to c'$ and maps $c_0 \leftarrow c_2 \to c_1$ making the resulting diagram commute. Given $c \leftarrow c_0 \to c'$ and $c' \leftarrow c_1 \to c''$, applying axiom a) to $c_0 \to c' \leftarrow c_1$ gives $c_0 \leftarrow c_2 \to c_1$, and the composition is $c \leftarrow c_0 \leftarrow c_2 \to c_1 \to c''$.

Equivalence:

$$\begin{tikzcd}
 & & c_2 \arrow[dl, dotted] \arrow[dr, dotted] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\
 & c_0 \arrow[dl] \arrow[drrr] & & c_1 \arrow[dlll] \arrow[dr] & \\
c & & & & c'
\end{tikzcd}$$

Composition:

$$\begin{tikzcd}
 & & c_2 \arrow[dl, dotted] \arrow[dr, dotted] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\
 & c_0 \arrow[dl] \arrow[dr] & & c_1 \arrow[dl] \arrow[dr] & \\
c & & c' & & c''
\end{tikzcd}$$