Fundamental groups in non-Archimedean geometry

CMI Mathematics Seminar

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I

The de Jong fundamental group

K: non-Archimedean field e.g. $k((t))$, $\mathbf{Q}_p$, $\mathbf{C}_p$

Tate: theory of rigid-analytic spaces over K

1. Tate algebra

$$K\langle x_1, \ldots, x_r \rangle = \left\{ \sum_{n \in \mathbf{N}^r} a_n \mathbf{x}^n \in K[[x_1, \ldots, x_r]] : a_n \to 0 \text{ as } |n| \to \infty \right\}$$

2. Affinoid K-algebras and affinoid spaces

$$X = \operatorname{Sp} A, \qquad A = K\langle x_1, \ldots, x_r \rangle / I$$

(underlying set = maximal ideals)

3. Glued together using the admissible topology

Nowadays we typically use adic spaces or Berkovich spaces

Definition (Berkovich '93, de Jong '95)

A morphism $Y \to X$ of rigid K-spaces is called a de Jong covering if there exists an overconvergent open cover $\{U_i\}$ of X such that Y_{U_i} is the disjoint union of finite étale coverings of U_i for all i.

- rigid K-space : adic space locally of finite type over K
- overconvergent open : open of X coming from $[X] = X^{\mathrm{Berk}}$ (a.k.a. *partially proper opens*, a.k.a. *wide open subsets*)

(e.g. $\{x : |f(x)| < 1\}$ is oc but $\{x : |f(x)| \leq 1\}$ is often not)

$\mathbf{Cov}_X^{\mathrm{oc}}$: the category of de Jong covering spaces of X.

$\mathbf{Cov}_X^{\mathrm{oc}}$ contains:

- $\mathbf{F\acute{E}t}_X = \{\text{Finite étale } X\text{-spaces}\}$
- $\mathbf{UF\acute{E}t}_X = \left\{\text{Disjoint unions of objects of } \mathbf{F\acute{E}t}_X\right\}$
- the category of 'topological coverings' of X
- André–Lepage's category of tempered coverings

Example (de Jong, J.K. Yu)

The Gross–Hopkins period map

$$\pi_{\mathrm{GH}} : \mathcal{M}^{\mathrm{LT}}_{\mathbf{C}_p} \to \mathbf{P}^{1,\mathrm{an}}_{\mathbf{C}_p}$$

is a de Jong covering.

$\mathbf{UCov}_X^{\mathrm{oc}}$: the category of disjoint unions of de Jong covering spaces of X.

Theorem (de Jong)

Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then,

$$(\mathbf{UCov}_X^{\mathrm{oc}}, F_{\overline{x}}), \qquad F_{\overline{x}}(Y) = Y_{\overline{x}}$$

is a tame infinite Galois category. In particular, if we set

$$\pi_1^{\mathrm{oc}}(X, \overline{x}) := \mathrm{Aut}(F_{\overline{x}})$$

then one obtains an equivalence of categories

$$F_{\overline{x}} : \mathbf{UCov}_X^{\mathrm{oc}} \xrightarrow{\sim} \pi_1^{\mathrm{oc}}(X, \overline{x})\text{-}\mathbf{Set}$$

$\pi_1^{\mathrm{oc}}(X, \overline{x})$: the de Jong fundamental group.

Property	de Jong covering space
closed under disjoint unions	no
closed under compositions	no
oc open local	yes
admissible local	???
étale local	???

$$\mathbf{Cov}_X^\tau = \left\{ \begin{matrix} Y \to X \text{ which } \tau\text{-locally on } X \text{ are the} \\ \text{disjoint union of finite étale coverings} \end{matrix} \right\} \qquad \tau \in \{\text{oc}, \text{adm}, \text{ét}\}$$

Question 1 (de Jong)

Are de Jong coverings admissible local on the target? I.e. $\mathbf{Cov}_X^{\text{oc}} = \mathbf{Cov}_X^{\text{adm}}$?

Question 2 (de Jong)

Is the pair $(\mathbf{UCov}_X^{\text{adm}}, F_{\overline{x}})$ a tame infinite Galois category?

Question 3

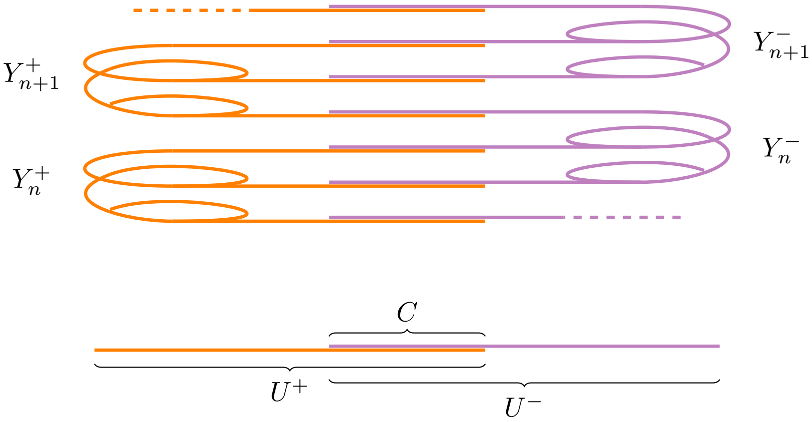
What about étale local on the target? What about $(\mathbf{UCov}_X^{\text{ét}}, F_{\overline{x}})$?

Theorem

Let K be a non-archimedean field of characteristic p, and let X be an annulus over K. Then, the containment $\mathbf{Cov}_X^{\mathrm{oc}} \subseteq \mathbf{Cov}_X^{\mathrm{adm}}$ is strict.

- $X = \{|\varpi| \leq |x| \leq |\varpi|^{-1}\}$
- $U^- = \{|\varpi| \leq |x| \leq 1\}, \qquad U^+ = \{1 \leq |x| \leq |\varpi|^{-1}\}$
- $C = U^- \cap U^+ = \{|x| = 1\}$

Idea of construction: The covering $Y \to X$ is obtained by gluing two families $Y_n^{\pm}$ of Artin–Schreier coverings of $U^{\pm}$ which are split over shrinking overconvergent neighborhoods of C.


Y_{n+1}^-
Y_{n+1}^+
Y_n^-
Y_n^+
C
U^+
U^-

Question

Do there exist examples in mixed characteristic? We are confident the answer is yes, and one can adapt the example in equicharacteristic p.

Theorem

Let K be a discretely valued non-archimedean field of equicharacteristic 0. Then, for any smooth X one has the equality $\mathbf{Cov}_X^{\text{oc}} = \mathbf{Cov}_X^{\text{adm}} = \mathbf{Cov}_X^{\text{ét}}$.

II

Geometric arcs
& geometric coverings

Independence of base point

De Jong's theory hinges on the following result ("existence of étale paths"):

Theorem (de Jong)

Let X be a connected rigid K-space and $\overline{x}$ and $\overline{y}$ geometric points of X. Then, $\pi_1^{\mathrm{oc}}(X;\overline{x},\overline{y})$ is non-empty.

$$\pi_1^{\mathcal{C}}(X;\overline{x},\overline{y}) = \mathrm{Isom}(F_{\overline{x}}|_{\mathcal{C}}, F_{\overline{y}}|_{\mathcal{C}}), \qquad \mathcal{C} \subseteq \mathbf{\acute{E}t}_X$$

$$\pi_1^{\mathbf{UF\acute{E}t}_X}(X;\overline{x},\overline{y}) = \pi_1^{\mathrm{alg}}(X;\overline{x},\overline{y}), \quad \text{algebraic étale paths}$$

$$\pi_1^{\mathbf{Cov}_X^{\tau}}(X;\overline{x},\overline{y}) = \pi_1^{\tau}(X;\overline{x},\overline{y}) \qquad \tau \in \{\mathrm{oc}, \mathrm{adm\,\acute{e}t}\}$$

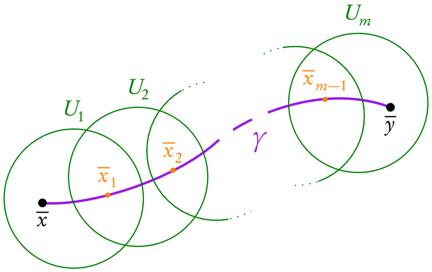
Independence of base point

γ arc connecting x and y in $[X]$

$\mathfrak{U} = \{U_1, \ldots, U_n\}$ "γ-nice" open cover

$\overline{x}_i \in \gamma \cap U_i \cap U_{i+1}$

$\mathbf{Cov}_{\mathfrak{U}}$: coverings split into finite étale over U_i



$$K_{\mathfrak{U}} = \mathrm{im}\left(\underbrace{\pi_1^{\mathrm{alg}}(U_1; \overline{x}, \overline{x}_1) \times \cdots \times \pi_1^{\mathrm{alg}}(U_m; \overline{x}_{m-1}, \overline{y})}_{\text{compact}} \to \pi_1^{\mathfrak{U}}(X; \overline{x}, \overline{y}) \right)$$

$$\emptyset \neq \varprojlim_{\mathfrak{U}} K_{\mathfrak{U}} \subseteq \varprojlim_{\mathfrak{U}} \pi_1^{\mathfrak{U}}(X; \overline{x}, \overline{y}) = \pi_1^{\mathrm{oc}}(X; \overline{x}, \overline{y}) \qquad \square$$

Geometric arc

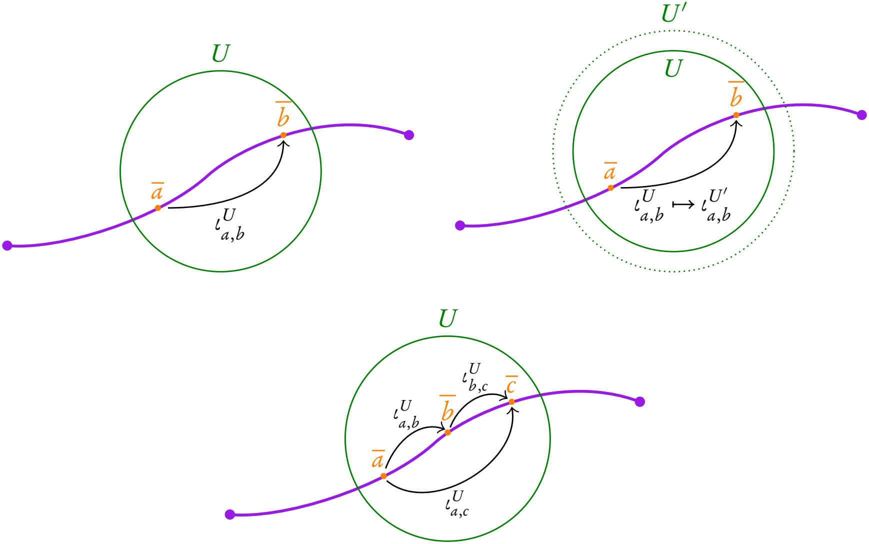
Definition

A geometric arc $\overline{\gamma}$ in X consists of:

- an arc γ in $[X]$,
- for every $z \in \gamma$, a geometric point $\overline{z}$ of X anchored at $z^{\max}$,
- for every subarc $[a,b] \subseteq \gamma$, and every open oc neighborhood U of $[a,b]$ an element $\iota_{a,b}^{U} \in \pi_1^{\mathrm{alg}}(U;\overline{a},\overline{b})$,

such that:

1. for all $[a,b] \subseteq \gamma$ and open oc neighborhoods $U \subseteq U'$ of $[a,b]$
$$\pi_1^{\mathrm{alg}}(U;\overline{a},\overline{b}) \to \pi_1^{\mathrm{alg}}(U';\overline{a},\overline{b}) \quad \text{maps} \quad \iota_{a,b}^{U} \mapsto \iota_{a,b}^{U'},$$
2. for every $U \subseteq X$ open oc neighborhood of $[a,c] = [a,b] \cup [b,c]$,
$$\pi_1^{\mathrm{alg}}(U;\overline{a},\overline{b}) \times \pi_1^{\mathrm{alg}}(U;\overline{b},\overline{c}) \to \pi_1^{\mathrm{alg}}(U;\overline{a},\overline{c}) \quad \text{maps} \quad (\iota_{a,b}^{U}, \iota_{b,c}^{U}) \mapsto \iota_{a,c}^{U}.$$


U
b̄
ā
ι^U_{a,b}
U′
U
b̄
ā
ι^U_{a,b} ↦ ι^{U′}_{a,b}
U
ι^U_{b,c}
c̄
ι^U_{a,b}
b̄
ā
ι^U_{a,c}

Geometric path connectedness

Theorem (de Jong, Berkovich, ALY)

Suppose X is connected and let x and y be maximal points of X. Then, there exists an extension L/K, smooth connected affinoid L-curves C_i, and maps $C_i \to X$ such that

1. $\mathrm{im}(C_i \to X) \cap \mathrm{im}(C_{i+1} \to X)$ *is non-empty,*
2. $x \in \mathrm{im}(C_1 \to X)$, *and* $y \in \mathrm{im}(C_m \to X)$

Theorem

Let X be a connected, smooth, and separated rigid K-curve. Then, for any two maximal geometric points $\overline{x}$ and $\overline{y}$ of X there exists a geometric arc $\overline{\gamma}$ that has $\overline{x}, \overline{y}$ as its endpoints.

Morally: connected rigid K-spaces are 'geometric path connected'.

Geometric coverings

Definition

A map $Y \to X$ satisfies unique lifting of geometric arcs if for all geometric arcs $\overline{\gamma}$ of X with left geometric endpoint $\overline{x}$, and every lift $\overline{x}'$ of $\overline{x}$, there exists a unique lift $\overline{\gamma}'$ of $\overline{\gamma}$ with left geometric endpoint $\overline{x}'$.

Definition

A morphism $Y \to X$ of rigid K-spaces is called a geometric covering if it is

1. étale,
2. partially proper,
3. and for all test curves $C \to X$ the map $Y_C \to C$ satisfies unique lifting of geometric arcs.

test curve : a map $C \to X$ over K where C is a smooth separated rigid L-curve for some extension L/K.

Properties of geometric coverings

$\mathbf{Cov}_X$: category of geometric coverings of X

Property	de Jong covering	geometric covering
disjoint unions	no	yes
composition	no	yes
oc open local	yes	yes
admissible local	no	yes
étale local	no	yes

$$\rightsquigarrow$$

$$\mathbf{Cov}_X^{\mathrm{oc}} \subseteq \mathbf{Cov}_X^{\mathrm{adm}} \subseteq \mathbf{Cov}_X^{\text{ét}} \subseteq \mathbf{Cov}_X$$

The geometric arc fundamental group

Theorem

Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then, $(\mathbf{Cov}_X, F_{\overline{x}})$ is a tame infinite Galois category. In particular, if we set

$$\pi_1^{\mathrm{ga}}(X, \overline{x}) := \mathrm{Aut}(F_{\overline{x}})$$

then we have an equivalence

$$F_{\overline{x}} : \mathbf{Cov}_X^{\mathrm{oc}} \xrightarrow{\sim} \pi_1^{\mathrm{ga}}(X, \overline{x})\text{-}\mathbf{Set}$$

$\pi_1^{\mathrm{oc}}(X, \overline{x})$: the geometric arc fundamental group.

Note: The non-emptiness of $\pi_1^{\mathrm{ga}}(X; \overline{x}, \overline{y})$ is now the easy part!

Answer to Question 2 and Question 3

Theorem

Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then, for $\tau \in \{\mathrm{adm}, \mathrm{\acute{e}t}\}$ the pair $(\mathbf{UCov}_X^\tau, F_{\overline{x}})$ is a tame infinite Galois category. In particular, if we set

$$\pi_1^\tau(X, \overline{x}) := \mathrm{Aut}(F_{\overline{x}})$$

then we get an equivalence

$$F_{\overline{x}} : \mathbf{UCov}_X^\tau \xrightarrow{\sim} \pi_1^\tau(X, \overline{x})\text{-}\mathbf{Set}$$

We get a series of maps of topological groups with dense image

$$\pi^{\mathrm{ga}}(X, \overline{x}) \to \pi_1^{\mathrm{\acute{e}t}}(X, \overline{x}) \to \pi_1^{\mathrm{adm}}(X, \overline{x}) \to \pi_1^{\mathrm{oc}}(X, \overline{x})$$

III

Relationship to Bhatt–Scholze's geometric coverings and AVC

Bhatt–Scholze's geometric coverings

Definition (Bhatt–Scholze)

Let X be a locally topologically Noetherian scheme. A morphism $Y \to X$ is a geometric covering if it is étale and partially proper (satisfies the valuative criterion of properness).

$\mathbf{Cov}_X$: the category of geometric coverings of X.

$\pi_1^{\text{proét}}(X, \overline{x})$: the fundamental group of the tame infinite Galois category $(\mathbf{Cov}_X, F_{\overline{x}})$ (Bhatt–Scholze).

Example

Let $\mathbf{D}_K$ be the closed unit disk, and $\mathbf{D}_K^\circ$ the open unit disk. Then, $\mathbf{D}_K^\circ \hookrightarrow \mathbf{D}_K$ is étale and partially proper.

AVC

Definition

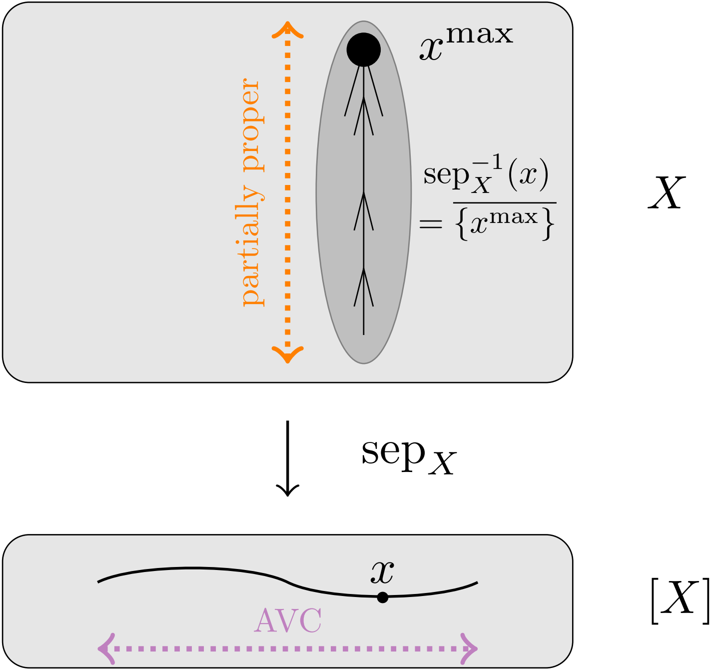
A map $Y \to X$ of rigid K-spaces satisfies the arcwise valuative criterion (AVC) if for every commutative square of solid arrows

$$\begin{array}{ccc} [0,1) & \longrightarrow & [Y] \\ \downarrow & \nearrow & \downarrow{\scriptstyle [f]} \\ [0,1] & \xrightarrow[i]{} & [X] \end{array}$$

where i is a topological embedding, there exists a unique dotted arrow making the diagram commute.

Example

$\mathbf{D}_K^\circ \hookrightarrow \mathbf{D}_K$ does not satisfy AVC.



$x^{\max}$
partially proper
$\mathrm{sep}_X^{-1}(x) = \overline{\{x^{\max}\}}$
X
sep_X
x
AVC
$[X]$

AVC and geometric coverings

Theorem

Let C be a smooth separated rigid K-curve. Then, for an étale and partially proper map $Y \to C$, the following are equivalent:

1. *$Y \to C$ satisfies unique lifting of geometric arcs,*
2. *$Y \to C$ satisfies AVC.*

Morally: A geometric covering is a map of rigid spaces which:

1. is étale,
2. satisfies a geometric valuative criterion (partial properness),
3. satisfies a topological valuative criterion (AVC).

Specialization map

Another, more literal, connection between geometric coverings in our sense and those of Bhatt–Scholze:

Theorem

Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme. Then, for any geometric point $\overline{x}$ of $\mathfrak{X}$ there is a specialization map

$$\pi_1^{\mathrm{oc}}(\mathfrak{X}_\eta, \overline{x}_\eta) \to \pi_1^{\mathrm{pro\acute{e}t}}(\mathfrak{X}_s, \overline{x}_s)$$

which has dense image if $\mathfrak{X}_s$ is reduced.

IV

Final thoughts

Further Questions

- Is there a theory of 'tame arcs' that would allow one to avoid restriction to curves?
- Is there a topology τ for which $\mathbf{Cov}_X = \mathbf{Loc}(X_\tau)$?
- Can geometric arcs be understood in terms of morphisms of topoi $\mathbf{Sh}([0,1]) \to \mathbf{Sh}(X^{\mathrm{Berk}})$? (suggested by Scholze)
- Can geometric arcs be used to study other things (e.g. exit paths and constructible sheaves)?

Thanks for listening!