Categorical local Langlands conjecture: refinements, questions, and speculations

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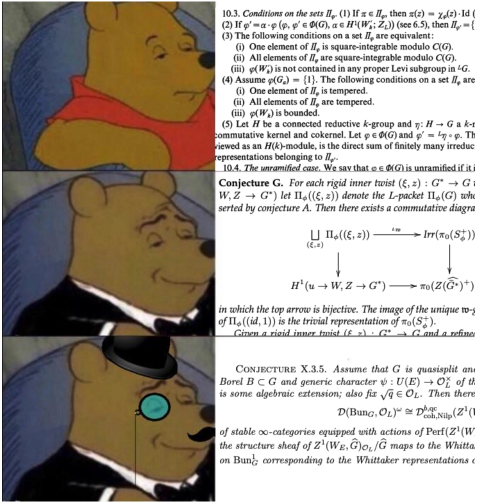
The talk in meme form

The local Langlands conjecture has evolved... a lot.



Fix $G/\mathbf{Q}_p$ a reductive group. The main geometric object in FS is the stack Bun_G of G-bundles on the Fargues-Fontaine curve.
Key points:

- Bun_G behaves like a smooth Artin stack of dimension zero.
- Natural bijection $|\mathrm{Bun}_G| = B(G) = G(\breve{\mathbf{Q}}_p)/\sigma - \mathrm{conj}$, with $b \in B(G) \rightsquigarrow \mathrm{Bun}_G^b \approx */G_b(\mathbf{Q}_p)$ locally closed substack. b basic $\leftrightarrow$ G_b an inner form of G $\leftrightarrow$ Bun_G^b open in Bun_G.
- Have a natural derived category of ℓ-adic sheaves $D(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$, with an infinite semi-orthogonal decomposition into categories $D(\mathrm{Bun}_G^b, \overline{\mathbf{Q}_\ell}) \cong D(G_b(\mathbf{Q}_p), \overline{\mathbf{Q}_\ell})$ reflecting the stratification.
- For each $b \in B(G)$, have adjoint functors $i_{b!} \vdash i_b^!$ and $i_{b\sharp} \vdash i_b^* \vdash i_{b*}$.
- Two natural finiteness conditions: ULA objects and compact objects. A ULA $\approx$ each $i_b^* A$ is a bounded complex of admissible representations. A compact $\approx$ A has finite support, and each $i_b^* A$ is a bounded complex of fin. gen. representations. $D(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$ is compactly generated.
- Two natural dualities: Verdier duality, and Bernstein-Zelevinsky duality. Verdier duality preserves ULA objects, and Bernstein-Zelevinsky duality preserves compact objects. Compatible with classical dualities on strata.

The essential carriers of information in the FS theory are the **Hecke operators**: these are certain endofunctors $T_V \circlearrowright D(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$, canonically associated with any $V \in \mathrm{Rep}({}^LG)$.

Key points:

- T_V has left and right adjoints, both given by $T_{V^\vee}$.
- Hecke operators preserve compact objects and ULA objects.
- $T_V \circ T_W \cong T_W \circ T_V \cong T_{V\otimes W}$ for all V, W.
- $T_V(A)$ carries a natural "continuous" $W_{\mathbf{Q}_p}$-action.

These formal properties are already enough for two major applications:
1. Finiteness theorems for the cohomology of local shtuka spaces.
2. Construction of canonical semisimple L-parameter $\varphi_\pi : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell})$ for any $\pi \in \mathrm{Irr}_{\overline{\mathbf{Q}_\ell}}(G(\mathbf{Q}_p))$.

However, FS also do something much deeper. For this we need to explicate the dual side a bit more.

Let ${}^L G = \hat{G} \rtimes W_{\mathbf{Q}_p}$ be the L-group, now regarded as a group scheme over $\mathbf{Z}_\ell$. The key statements on the dual side are the following, due independently to Fargues-Scholze, Dat-Helm-Kurinczuk-Moss, and Zhu.

- There is a natural moduli space $Z^1(W_{\mathbf{Q}_p}, \hat{G})$ parametrizing ℓ-adically continuous L-parameters $\varphi : W_{\mathbf{Q}_p} \to {}^L G$, which is a disjoint union of finite type flat lci $\mathbf{Z}_\ell$-schemes of pure relative dimension $\dim G$.
- The stack quotient $\mathrm{Par}_G = Z^1(W_{\mathbf{Q}_p}, \hat{G})/\hat{G}$ is a disjoint union of finite type flat Artin stacks, relative lci over $\mathbf{Z}_\ell$, of pure relative dimension zero.
- The coarse quotient $Z^1(W_{\mathbf{Q}_p}, \hat{G})//\hat{G}$ is a disjoint union of finite type flat $\mathbf{Z}_\ell$-schemes. For algebraically closed fields $L/\mathbf{Z}_\ell$, points $\mathrm{Spec}\, L \to Z^1(W_{\mathbf{Q}_p}, \hat{G})//\hat{G}$ naturally correspond to isomorphism classes of semisimple L-parameters $W_{\mathbf{Q}_p} \to {}^L G(L)$.
- The ring of regular functions $\mathcal{Z}^{\mathrm{spec}}(G) = \mathcal{O}(\mathrm{Par}_G) = \mathcal{O}(Z^1(W_{\mathbf{Q}_p}, \hat{G})//\hat{G})$ is the "spectral Bernstein center".

Can repeat all of these constructions after base change to any $\mathbf{Z}_\ell$-algebra Λ.

The most subtle result in FS is surely the following.

Theorem

Let L be an algebraic extension of $\mathbf{Q}_\ell(\sqrt{p})$, and let $\Lambda \in \{L, \mathcal{O}_L\}$. Assume that $\Lambda = L$ or that ℓ is a very good prime for G. Then there is a canonical Λ-linear $\otimes$-action of $\mathrm{Perf}(\mathrm{Par}_{G,\Lambda})$ on $D(\mathrm{Bun}_G, \Lambda)$, extending the action of Hecke operators and preserving the subcategory of compact objects.

When $\Lambda = L$ this is not so hard, following ideas of AGKRRV, but the integral case is very hard. FS then suggest the following deep and beautiful conjecture.

Conjecture

Let Λ be as in the previous theorem, and containing all p-power roots of unity. Assume that G is quasisplit, and choose a Whittaker datum $(B, \psi : U_B(\mathbf{Q}_p) \to \Lambda^\times)$. Then there is a canonical equivalence of categories

$$\mathbf{L}^G_\psi : D^{b,\mathrm{qc}}_{\mathrm{coh},\mathrm{Nilp}}(\mathrm{Par}_{G,\Lambda}) \xrightarrow{\sim} D(\mathrm{Bun}_G, \Lambda)^{\mathrm{cpct}}$$

compatible with the spectral action, and (after ind-extension) sending $\mathcal{O}_{\mathrm{Par}_{G,\Lambda}}$ to $i_{\mathbf{1}!} W_\psi$.

This requires some explanations.

- $W_\psi = \mathrm{c}-\mathrm{ind}_{U_B(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}\psi$ is the space of compactly supported Whittaker functions with coefficients in Λ. (Makes sense for Λ any $\mathbf{Z}[p^{-1},\zeta_{p^\infty}]$-algebra.)
- "Compatible with the spectral action" means that for $A \in \mathrm{Perf}(\mathrm{Par}_{G,\Lambda})$ and $B \in D^{b,\mathrm{qc}}_{\mathrm{coh,Nilp}}(\mathrm{Par}_{G,\Lambda})$, we should have $\mathbf{L}_\psi^G(A \otimes B) \simeq A * \mathbf{L}_\psi^G(B)$.
- The subscript "Nilp" refers to the condition of **nilpotent singular support** (Arinkin-Gaitsgory). Automatic if $\Lambda = L$ or ℓ is sufficiently large. In general, have $\mathrm{Perf}^{\mathrm{qc}}(\mathrm{Par}_{G,\Lambda}) \subset D^{b,\mathrm{qc}}_{\mathrm{coh,Nilp}}(\mathrm{Par}_{G,\Lambda})$.

Taking all these observations together, we see that $\mathbf{L}_\psi^G$ restricted to $\mathrm{Perf}^{\mathrm{qc}}(\mathrm{Par}_{G,\Lambda})$ should be given by the functor $C \mapsto C * i_{1!}W_\psi$.
Empirical observation: Meditating on the categorical conjecture is a good way of generating new conjectures!
Zeroth example: The categorical conjecture implicitly demands that for any $C \in \mathrm{Perf}^{\mathrm{qc}}(\mathrm{Par}_{G,\Lambda})$, $C * i_{1!}W_\psi$ is compact. True for $G = \mathrm{GL}_n$, but an open problem in general! Closely related to showing that $\pi \mapsto \varphi_\pi$ has finite fibers.

First example

What does the full faithfulness of $\mathbf{L}_\psi^G$ imply, when applied to the structure sheaf on $\mathrm{Par}_{G,\Lambda}$? Conjecture says that $\mathbf{L}_\psi^G(\mathcal{O}_{\mathrm{Par}_{G,\Lambda}}) \simeq i_{\mathbf{1}!}W_\psi$ so using also full faithfulness of $i_{\mathbf{1}!}$, we see that there should be a natural isomorphism

$$R\Gamma(\mathrm{Par}_{G,\Lambda}, \mathcal{O}_{\mathrm{Par}_{G,\Lambda}}) \simeq R\mathrm{End}(W_\psi)$$

of (derived?) rings.

Theorem (Fargues-Scholze)

Assume $\Lambda = L$ or that ℓ is very good. Then $H^*(\mathrm{Par}_{G,\Lambda}, \mathcal{O}_{\mathrm{Par}_{G,\Lambda}})$ vanishes outside degree zero, and $H^0 = \mathcal{O}(Z^1(W_{\mathbf{Q}_p}, \hat{G})_\Lambda)^{\hat{G}}$ is a countable product of finite type flat Λ-algebras.

This immediately translates into a concrete conjecture on the other side!

Conjecture

Notation as above, $R\mathrm{End}(W_\psi)$ vanishes outside degree zero, and $End(W_\psi)$ is a countable product of finite type flat (commutative!) Λ-algebras. Moreover, the ring $End(W_\psi)$ is **canonically independent** *of the choice of Whittaker datum.*

Theorem (H.)

Let Λ be any $\mathbf{Z}[p^{-1}, \zeta_{p^\infty}]$-algbra, and pick a Whittaker datum (B, ψ), with W_ψ as before.

1. *W_ψ is a* **projective** *object.* $\rightsquigarrow$ $R\mathrm{End}(W_\psi)$ *vanishes outside degree zero, and $End(W_\psi)$ is flat over Λ.*
2. *For $\Lambda \in \{L, \mathcal{O}_L\}$ as in the categorical conjecture, the ring $End(W_\psi)$ is a countable product of (commutative) finite type flat Λ-algebras.*

This theorem was also proved independently by Dat-Helm-Kurinczuk-Moss.
When $\Lambda = \mathbf{C}$, 1. is due to Chan-Savin, and 2. is due to Bushnell-Henniart (in a more precise form).
Proof uses an approximation technique of Rodier together with BH's results.
Still open: The ring $End(W_\psi)$ should be canonically independent of the choice of Whittaker datum. No idea how to prove this!
Small evidence: Can prove that $End(W_\psi) \simeq End(W_{\psi^{-1}})$.

Conjecture. For any parabolic $P = MU \subset G$ and Λ any $\mathbf{Z}_\ell[\sqrt{p}]$-algebra, there is a natural functor $\mathrm{Eis}_P = \mathrm{Eis}_P^G : D(\mathrm{Bun}_M, \Lambda) \to D(\mathrm{Bun}_G, \Lambda)$ with the following properties.

1. There is a natural equivalence $\mathrm{Eis}_P^G \circ i_{1!} \simeq i_{1!} \circ \mathrm{Ind}_P^G$, where $\mathrm{Ind}_P^G : D(M(\mathbf{Q}_p), \Lambda) \to D(G(\mathbf{Q}_p), \Lambda)$ is normalized parabolic induction.
2. Compatibility with composition: For any $P_1 = M_1 U_1 \subset P_2 = M_2 U_2$, $P_1 \cap M_2$ is a parabolic in M_2 with Levi M_1, and should have an equivalence $\mathrm{Eis}_{P_1}^G \simeq \mathrm{Eis}_{P_2}^G \circ \mathrm{Eis}_{P_1 \cap M_2}^{M_2}$.
3. Eis_P is compatible with any extension of scalars $\Lambda \to \Lambda'$.
4. Eis_P commutes with direct sums.
5. Eis_P preserves compact objects, and ULA objects with quasicompact support.
6. When Λ is killed by a power of ℓ, Eis_P **is** the functor $p_!(\mathrm{IC}_{\mathrm{Bun}_P} \otimes q^*(-))$, where $\mathrm{Bun}_M \xleftarrow{q} \mathrm{Bun}_P \xrightarrow{p} \mathrm{Bun}_G$ is the usual diagram, and $\mathrm{IC}_{\mathrm{Bun}_P}$ is a certain (explicit) invertible object in $D(\mathrm{Bun}_P, \Lambda)$.

Of course, in the torsion coefficients case, part 6. gives a definition of Eis_P, and then proving 1.-5. is a definite task. 1.-4. are easy, but 5. seems much harder.

Now fix Λ as in the categorical conjecture. On the dual side, we can (unconditionally!) define a similar functor

$$\mathrm{Eis}_P^{\mathrm{spec}} = p_*^{\mathrm{spec}} q^{\mathrm{spec}*} : D^{b,\mathrm{qc}}_{\mathrm{coh},\mathrm{Nilp}}(\mathrm{Par}_{M,\Lambda}) \to D^{b,\mathrm{qc}}_{\mathrm{coh},\mathrm{Nilp}}(\mathrm{Par}_{G,\Lambda}),$$

where $\mathrm{Par}_{M,\Lambda} \xleftarrow{q^{\mathrm{spec}}} \mathrm{Par}_{P,\Lambda} \xrightarrow{p^{\mathrm{spec}}} \mathrm{Par}_{G,\Lambda}$ is the analogous diagram. Preservation of "Nilp" is formal (argument already in Arinkin-Gaitsgory).

Conjecture

Assume G is quasisplit, and fix (B,ψ) as before. Then for $P = MU \subset G$ any semi-standard parabolic, there should be a natural equivalence of functors

$$\mathbf{L}_\psi^G \circ \mathrm{Eis}_P^{\mathrm{spec}} \simeq \mathrm{Eis}_P \circ \mathbf{L}_{\psi_M}^M .$$

Motivated by naive analogy with "classical" geometric Langlands.
We will see that this conjecture suggests several more conjectures purely on the automorphic side, which pass various sanity checks.

How does Eis_P interact with Hecke operators? Hard question. Easier question: How does $\mathrm{Eis}_P^{\mathrm{spec}}$ interact with Hecke operators?
What does this even mean? Let us assume for simplicity that G is **split**. Have a tautological map $\tau_G : \mathrm{Par}_{G,\Lambda} \to B\hat{G}_\Lambda$.
Basic fact: for any $V \in \mathrm{Rep}(\hat{G}_\Lambda) \subset \mathrm{Perf}(B\hat{G}_\Lambda)$, the spectral action $\tau_G^* V * (-) \circlearrowright D(\mathrm{Bun}_G, \Lambda)$ **is** the Hecke operator $T_V(-)$.
$\rightsquigarrow$ By compatibility of $\mathbf{L}_\psi^G(-)$ with the spectral action, get $T_V \circ \mathbf{L}_\psi^G(-) \simeq \mathbf{L}_\psi^G \circ (\tau_G^* V \otimes -)$.
So $T_V \circ \mathrm{Eis}_P$ corresponds to $\tau_G^* V \otimes \mathrm{Eis}_P^{\mathrm{spec}}$. Can we rewrite this latter thing in some enlightening way?

Exercise: Choose a finite filtration $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_m = V|\hat{P}_\Lambda$ such that $\hat{U}_\Lambda$ acts trivially on each $W_i = V_i/V_{i-1}$, i.e. such that W_i is naturally inflated from $\mathrm{Rep}(\hat{M}_\Lambda)$. Then $\tau_G^* V \otimes \mathrm{Eis}_P^{\mathrm{spec}}(-)$ admits a natural finite filtration with graded pieces $\mathrm{Eis}_P^{\mathrm{spec}}(\tau_M^* W_i \otimes -)$.

Sketch: Look at the diagram

$$\begin{tikzcd}
\mathrm{Par}_{M,\Lambda} \arrow[d, "\tau_M"] & \mathrm{Par}_{P,\Lambda} \arrow[l, "q^{\mathrm{spec}}"'] \arrow[r, "p^{\mathrm{spec}}"] \arrow[d, "\tau_P"] & \mathrm{Par}_{G,\Lambda} \arrow[d, "\tau_G"] \\
B\hat{M}_\Lambda & B\hat{P}_\Lambda \arrow[l] \arrow[r] & B\hat{G}_\Lambda
\end{tikzcd}$$

and use the projection formula to write $\tau_G^* V \otimes \mathrm{Eis}_P^{\mathrm{spec}}(-) = p_*^{\mathrm{spec}}(\tau_P^*(V|\hat{P}_\Lambda) \otimes q^{\mathrm{spec}*}(-))$. Then $\tau_P^*(V|\hat{P}_\Lambda)$ has a filtration with graded pieces $q^{\mathrm{spec}*}\tau_M^* W_i$, and $q^{\mathrm{spec}*}$ is symmetric monoidal. But now we can turn this into a conjecture on the automorphic side!

To repeat: Choose a finite filtration $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_m = V|\hat{P}_\Lambda$ such that $\hat{U}_\Lambda$ acts trivially on each $W_i = V_i/V_{i-1}$, i.e. such that W_i is naturally inflated from $\mathrm{Rep}(\hat{M}_\Lambda)$. Then $\tau_G^* V \otimes \mathrm{Eis}_P^{\mathrm{spec}}(-)$ admits a natural finite filtration with graded pieces $\mathrm{Eis}_P^{\mathrm{spec}}(\tau_M^* W_i \otimes -)$.
Translating back to the automorphic side, we get:

Conjecture

Notation and choices as above, the functor $T_V \circ \mathrm{Eis}_P(-)$ has a natural finite filtration with graded pieces $\mathrm{Eis}_P \circ T_{W_i}(-)$.

This conjecture makes sense for any Λ. When Λ is a torsion ring, this can be **proved** (H.-Scholze).

Duality

How does the categorical conjecture interact with duality? On $D(\mathrm{Bun}_G, \Lambda)$, have BZ duality and Verdier duality. The categorical equivalence should describe compact objects, and these are preserved by BZ duality. So we can ask: can $\mathbf{D}_{\mathrm{BZ}} \circ \mathbf{L}_{\psi}^{G}$ be rewritten in terms of some duality on the other side? On $D^b_{\mathrm{coh}}(\mathrm{Par}_{G,\Lambda})$, have Grothendieck-Serre duality. Very clean: since $\mathrm{Par}_{G,\Lambda}$ is an lci Artin stack of dimension zero, the GS duality functor is just $R\mathscr{H}\mathrm{om}(-, \mathcal{O}_{\mathrm{Par}_{G,\Lambda}})$.

Now, there is an additional symmetry on $\mathrm{Par}_{G,\Lambda}$: the Chevalley involution on $\hat{G}$ induces an order two automorphism i_{Ch} of $\mathrm{Par}_{G,\Lambda}$. Set $\mathbf{D}_{\mathrm{twGS}}(-) = i^*_{\mathrm{Ch}} R\mathscr{H}\mathrm{om}(-, \mathcal{O}_{\mathrm{Par}_{G,\Lambda}})$. "Twisted Grothendieck-Serre duality." Easy to see that i^*_{Ch} commutes with GS duality, so $\mathbf{D}^2_{\mathrm{twGS}} = \mathrm{id}$ on $D^b_{\mathrm{coh}}(\mathrm{Par}_{G,\Lambda})$. Preserves "Nilp" and "qc" conditions, and preserves Perf.

Conjecture

Notation as before, there is a natural equivalence of functors

$$\mathbf{D}_{\mathrm{BZ}} \circ \mathbf{L}_{\psi}^{G} \simeq \mathbf{L}_{\psi^{-1}}^{G} \circ \mathbf{D}_{\mathrm{twGS}}(-).$$

Again, this suggests several more conjectures!

Using the expected compatibility of $\mathbf{L}_\psi^G$ with the spectral action, this immediately suggests the following:

Conjecture

For all $C \in \mathrm{Perf}(\mathrm{Par}_{G,\Lambda})$ *and* $A \in D(\mathrm{Bun}_G, \Lambda)^{\mathrm{cpct}}$, *there is a natural isomorphism*

$$\mathbf{D}_{\mathrm{BZ}}(C * A) \simeq \mathbf{D}_{\mathrm{twGS}}(C) * \mathbf{D}_{\mathrm{BZ}}(A).$$

This conjecture is probably within reach! True when $C = \tau_G^* V$.
We can also start combining our expectations in more artful ways. For instance, recall that we conjectured $\mathbf{L}_\psi^G \circ \mathrm{Eis}_P^{\mathrm{spec}} \simeq \mathrm{Eis}_P \circ \mathbf{L}_{\psi_M}^M$. How does this interact with duality?

Duality and Eisenstein series

What happens if we dualize $\mathbf{L}^G_\psi \circ \mathrm{Eis}^{\mathrm{spec}}_P \simeq \mathrm{Eis}_P \circ \mathbf{L}^M_{\psi_M}$? We compute as follows:

$$\begin{aligned}\mathbf{D}_{\mathrm{BZ}} \circ \mathrm{Eis}_P \circ \mathbf{L}^M_{\psi_M} &\simeq \mathbf{D}_{\mathrm{BZ}} \circ \mathbf{L}^G_\psi \circ \mathrm{Eis}^{\mathrm{spec}}_P \\ &\simeq \mathbf{L}^G_{\psi^{-1}} \circ \mathbf{D}_{\mathrm{twGS}} \circ \mathrm{Eis}^{\mathrm{spec}}_P \\ &\overset{!}{\simeq} \mathbf{L}^G_{\psi^{-1}} \circ \mathrm{Eis}^{\mathrm{spec}}_{\overline{P}} \circ \mathbf{D}^M_{\mathrm{twGS}} \\ &\simeq \mathrm{Eis}_{\overline{P}} \circ \mathbf{L}^M_{\psi_M^{-1}} \circ \mathbf{D}^M_{\mathrm{twGS}} \\ &\simeq \mathrm{Eis}_{\overline{P}} \circ \mathbf{D}^M_{\mathrm{BZ}} \circ \mathbf{L}^M_{\psi_M}.\end{aligned}$$

But now we can cancel out $\mathbf{L}^M_{\psi_M}$ from the first and last equations, getting the following conjecture:

Conjecture

Notation as above, there is a natural equivalence

$$\mathbf{D}_{\mathrm{BZ}} \circ \mathrm{Eis}_P \simeq \mathrm{Eis}_{\overline{P}} \circ \mathbf{D}^M_{\mathrm{BZ}}.$$

Again, purely an automorphic statement! Moreover, using compatibility of Eis_P with parabolic induction, easy to see that this conjecture implies Bernstein's "second adjointness"!! (Recently proved with general coefficients by Dat-Helm-Kurinczuk-Moss.)

Some final speculations

Let me now be slightly vague.

Conjecture

Generically over the coarse moduli space, the functor Eis_P *depends* **only** *on the Levi* $M \subset G$*, and not on the choice of parabolic containing* M*.*

Again, the analogous statement for spectral Eisenstein series is easy.
Observation (H.-Scholze): This conjecture together with the earlier "$T_V \circ \mathrm{Eis}_P \overset{gr}{=} \bigoplus_i \mathrm{Eis}_P \circ T_{W_i}$" conjecture **implies** the Harris-Viehmann conjecture.
One more question: How to describe $\mathbf{L}_{\psi'}^{G,-1} \circ \mathbf{L}_{\psi}^{G}$ for two different Whittaker data (B, ψ) and (B', ψ')? This is some (non-monoidal) self-equivalence of $D^{b,\mathrm{qc}}_{\mathrm{coh},\mathrm{Nilp}}(\mathrm{Par}_{G,\Lambda})$. There should be a precise conjecture describing it in simple terms...

Thank you for listening!