ON DELIGNE'S MONODROMY FINITENESS THEOREM (MIMUW AG SEMINAR, DEC 13, 2018)

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ABSTRACT. Let S be a smooth complex algebraic variety with a base point 0. If X/S is a family of smooth projective varieties, one obtains the associated monodromy representation $\pi_1(S,0) \to \mathrm{GL}(H^*(X_0,\mathbf{Q}))$ of the topological fundamental group of S on the rational cohomology of X_0. Let us say that a representation $\pi_1(S,0) \to \mathrm{GL}_N(\mathbf{Q})$ "comes from geometry" if it is a direct summand of such a monodromy representation. Representations coming from geometry are quite special: they factor through $\mathrm{GL}_N(\mathbf{Z})$ up to conjugation, their monodromy groups are semisimple, and the corresponding local system on S underlies a polarizable variation of Hodge structures. Inspired by a theorem of Faltings, Deligne proved in the mid-80s that there are only finitely many such representations of rank N for fixed S and N. We explain the beautifully simple proof of this result, based on Griffiths' study of the geometry of period domains. We also discuss the very recent results of Litt which in particular imply a suitable analog for the étale fundamental group.

1. Monodromy representations

Ehresmann's theorem in differential geometry states that if

$$f\colon X \longrightarrow S$$

is a proper submersion of smooth manifolds, then f is a fiber bundle. Consequently, the cohomology groups of the fibers $H^n(X_s,\mathbf{Q})$ ($s\in S$) assemble into a local system $R^n f_*\mathbf{Q}$ on S. If S is connected and $0\in S$, such a local system corresponds to a representation

$$\sigma\colon \pi_1(S,0) \longrightarrow \mathrm{GL}(H^n(X_0,\mathbf{Q})),$$

called the *monodromy representation* associated to $f\colon X\to S$ (in degree n).

Let now S be a smooth complex algebraic variety, and let $f\colon X\to S$ be a smooth projective morphism. Then the fibers of f are not necessarily biholomorphic to one another, and the variation of the complex structure on the fibers is controlled by Kodaira–Spencer theory. However, the induced map of complex manifolds

$$f^{\mathrm{an}}\colon X^{\mathrm{an}} \longrightarrow S^{\mathrm{an}}$$

is a proper submersion, and one gets the associated monodromy representation

$$\sigma\colon \pi_1(S^{\mathrm{an}},0) \longrightarrow \mathrm{GL}(H^n(X_0^{\mathrm{an}},\mathbf{Q})).$$

Definition 1.1. We say that a representation $\sigma\colon \pi_1(S^{\mathrm{an}},0)\to GL_N(\mathbf{Q})$ *comes from geometry* if it is isomorphic to a subquotient (equivalently, a direct summand) of a monodromy representation attached to some smooth and projective $f\colon X\to S$.

The aim of this talk is to discuss the proof of the following beautiful result of Deligne.

Theorem 1 (Deligne 1987, [2])**.** *For fixed $(S, 0)$ and $N \geq 0$, there exist only finitely many isomorphism classes of representations $\sigma\colon \pi_1(S^{\mathrm{an}}, 0) \to GL_N(\mathbf{Q})$ which come from geometry.*

What are the properties of representations coming from geometry which make them special (and make the above theorem true)?

(i) Since $\pi_1(S^{\mathrm{an}}, 0) \to \mathrm{GL}(H^n(X^{\mathrm{an}}, \mathbf{Q}))$ factors through $\mathrm{GL}(H^n(X^{\mathrm{an}}, \mathbf{Z})/\mathrm{tors.})$, a representation $\sigma\colon \pi_1(S^{\mathrm{an}}, 0) \to GL_N(\mathbf{Q})$ coming from geometry has to be isomorphic to one which factors through $\mathrm{GL}_N(\mathbf{Z})$ (that is, σ *preserves a lattice*).

(ii) The Zariski closure of the image of $\pi_1(S^{\mathrm{an}}, 0) \to \mathrm{GL}(H^n(X^{\mathrm{an}}, \mathbf{Q}))$ is a *semisimple* algebraic group over $\mathbf{Q}$ [3]. Consequently, the full subcategory of representations spanned by those coming from geometry is semisimple (and in particular one can replace 'subquotient' with 'direct summand' in the definition).

(iii) The local system $R^n f_* \mathbf{Q}$ underlies a *polarized variation of Hodge structure* (we will explain what this is in §3), and so does every local system associated to a representation coming from geometry.

Theorem 1 will be deduced from a more general result in complex geometry:

Theorem 2. *Let $(S, 0)$ be a pointed connected complex manifold such that $\pi_1(S, 0)$ is finitely generated, and let $N \geq 0$.*

(a) There exist only finitely many isomorphism classes of $\mathbf{Q}$-local systems of rank N on S underlying a polarizable integral variation of Hodge structures, up to semisimplification.

(b) Suppose that S is compactifiable*, i.e. that there exists a compact complex manifold $\overline{S}$ and a closed analytic subset $Z \subseteq \overline{S}$ such that $S \simeq \overline{S} \setminus Z$. Then there exist only finitely many isomorphism classes of $\mathbf{Q}$-local systems of rank N on S which are a subquotient of a local system underlying a polarizable integral variation of Hodge structures.*

Here in (a), 'up to semisimplification' means that we identify two local systems if both can be equipped with filtrations such that the associated graded objects are isomorphic. In more concrete terms, this means that the associated characters (traces) are equal. And indeed this is how the theorem is proved: one proves a bound on these traces which depends only on N; since they are integral, they can take only finitely many values. Under the assumptions in (b), in fact the category of such subquotients is semisimple, so equality of traces implies isomorphism.

In Section 2, we discuss Hodge structures and period domains. In Section 3, we discuss variations of Hodge structures, period maps, and formulate Deligne's semi-simplicity theorem. In Section 4, we prove Theorems 1 and 2. We use local systems coming from families of elliptic curves as our running example.

In the last Section 5, we discuss a recent result of Litt [6] which provides analogs of the above results for the étale fundamental group.

2. Hodge structures

We will now review some standard material from Hodge theory. We refer to Voisin's book [8] for details.

2.1. **Hodge decomposition and Hodge structures.** Let X be a smooth and projective complex algebraic variety. Then the cohomology groups $H^n(X, \mathbf{C})$[1] have a canonical *Hodge decomposition*

$$H^n(X,\mathbf{C}) \cong \bigoplus_{p+q=n} H^{p,q}, \quad H^{p,q} = H^q(X, \Omega^p_X). \tag{2.1.1}$$

Since $H^n(X,\mathbf{C}) = H^n(X,\mathbf{R})\otimes\mathbf{C}$, the group $\mathrm{Gal}(\mathbf{C}/\mathbf{R}) \simeq \mathbf{Z}/2\mathbf{Z}$ acts on $H^n(X,\mathbf{C})$. Under this action, one has $\overline{H^{p,q}} = H^{q,p}$. For reasons soon to become apparent, it is more convenient to consider the *Hodge filtration*

$$F^i H^n(X,\mathbf{C}) = \bigoplus_{p\geq i} H^{p,q} = \mathrm{im}\left(H^n(X, \Omega_X^{\bullet \geq i}) \to H^n(X, \Omega_X^\bullet)\right);$$

one has $H^{p,q} = F^p \cap \overline{F^q}$ and $H^n(X,\mathbf{C}) = F^p \oplus \overline{F^{q+1}}$. This defines (2.1.1).

We say that the above equips $H^n(X,\mathbf{Z})/\text{tors.}$ with an integral Hodge structure in the sense of the definition below.

Definition 2.2. Let n be an integer.

(1) An *integral Hodge structure of weight* n is a finitely generated free $\mathbf{Z}$-module V together with a decomposition

$$V \otimes \mathbf{C} = \bigoplus_{p+q=n} V^{p,q}$$

satisfying $\overline{V^{p,q}} = V^{q,p}$. Equivalently, $V \otimes \mathbf{C}$ is equipped with a finite descending filtration F^p satisfying $F^p \oplus \overline{F^{q+1}} = V \otimes \mathbf{C}$.

(2) The numbers $h^{p,q} = \dim V^{p,q}$ are the *Hodge numbers* of V.

2.3. **Polarizations and Hard Lefschetz.** Let now $\omega \in H^1(X, \Omega^1_X)$ be the image of an ample class L under

$$d\log\colon \mathrm{Pic}(X) = H^1(X, \mathscr{O}_X^\times) \longrightarrow H^1(X, \Omega^1_X) \cap H^2(X,\mathbf{Z}).$$

The Hard Lefschetz theorem states that for $n \leq d = \dim X$ the cup product map

$$\omega^{d-n}\colon H^n(X,\mathbf{C}) \xrightarrow{\sim} H^{2d-n}(X,\mathbf{C}),$$

is an isomorphism. By basic linear algebra, this induces the *Lefschetz decomposition*

$$H^n(X,\mathbf{Q}) = \bigoplus_{k\geq 0} \omega^k \cdot H^{n-2k}_{\mathrm{prim}}(X,\mathbf{Q}),$$

where $H^r(X,\mathbf{C})_{\mathrm{prim}}$ is the *primitive cohomology*

$$H^r(X,\mathbf{C})_{\mathrm{prim}} = \ker\left(\omega^{d-r+1}\colon H^r(X,\mathbf{C}) \to H^{2d-r}(X,\mathbf{C})\right)$$

(for $r > d$ we set this to be zero). This decomposition is compatible with Hodge structures because ω is of type $(1,1)$.

Consider the bilinear *intersection pairing*

$$Q(\alpha,\beta) = \langle \omega^{d-n}\cdot\alpha, \beta\rangle = \frac{1}{(2\pi i)^n}\int_X \omega^{d-n}\wedge\alpha\wedge\beta,$$

on $H^n(X,\mathbf{C})$ which is symmetric for n odd and alternating for n even, and takes integral values for $\alpha,\beta \in H^n(X,\mathbf{Z})$. The associated hermitian pairing

$$H(\alpha,\beta) = (2\pi i)^n Q(\alpha,\overline{\beta}) = \int_X \omega^{d-n}\wedge\alpha\wedge\overline{\beta},$$

[1]We will suppress the notation X^{an} from now on.

has the property that the decomposition (2.1.1) is orthogonal with respect to H. Moreover, one has the *Hodge–Riemann bilinear relation*

$$(-1)^{\varepsilon} H(\alpha,\alpha) > 0 \quad \text{for} \quad 0 \neq \alpha \in H^{p,q} \cap H^n(X, \mathbf{C})_{\mathrm{prim}},$$

where $\varepsilon = q + n(n-1)/2$. The hermitian pairing equal to $(-1)^{k(k-1)/2+q+r} H$ on $\omega^r \cdot H^{n-2r}(X, \mathbf{C})_{\mathrm{prim}}$ is thus positive-definite.

Definition 2.4. Let n be an integer.

(1) A *polarized integral Hodge structure of weight* n is an integral Hodge structure V of weight n endowed with a bilinear pairing $Q(\alpha,\beta)\colon V \times V \to \mathbf{Z}$ such that the associated hermitian form $H(\alpha,\beta) = i^n Q(\alpha, \overline{\beta})$ on $V \otimes \mathbf{C}$ satisfies
   i. $H(V^{p,q}, V^{p',q'}) = 0$ if $p \neq p'$,
   ii. $(-1)^{\varepsilon} H(\alpha,\alpha) > 0$ for $0 \neq \alpha \in V^{p,q}$, where $\varepsilon = q + n(n-1)/2$.

(2) An integral Hodge structure V of weight n is *polarizable* if it admits a bilinear pairing as above.

Example 2.5. Let X be an elliptic curve. Then $H^1(X, \mathbf{Q}) = H^1(X, \mathbf{Q})_{\mathrm{prim}}$ has Hodge numbers $h^{1,0} = 1 = h^{0,1}$. Fix an isomorphism $H^1(X, \mathbf{Z}) \simeq \mathbf{Z}^2$ in which the intersection form Q is the standard symplectic matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The Hodge structure on $H^1(X, \mathbf{Q})$ amounts to the choice of a line $F^1 \subseteq \mathbf{C}^2$ such that $F^1 \cap \overline{F^1} = 0$, i.e. such that the associated point in $\mathbf{P}^1(\mathbf{C})$ does not lie in $\mathbf{P}^1(\mathbf{R})$. Thus F^1 can be given by an equation $v = \tau u$ where $\tau \in \mathbf{C} \setminus \mathbf{R}$. This Hodge structure is polarized by the pairing Q if and only if τ lies in the upper half-plane.

2.6. **Period domains.** The point of studying Hodge structures this is that while $H^n(X, \mathbf{Z})$ carries only topological information (e.g. the intersection form), the associated Hodge structure is interesting enough to distinguish between different varieties in a family. General 'Torelli theorems' assert that for certain classes of varieties (curves, abelian varieties, K3 surfaces, Calabi–Yau, hyperkähler, ...) the Hodge structure on the cohomology (possibly with additional data) determines the variety uniquely. Thinking in terms of moduli, we can think of the space D parametrizing polarized Hodge structures of weight n on a fixed (V, Q):

Definition 2.7. Let V be a real vector space of finite dimension and let n be an integer. Let H be a hermitian pairing on V, and let $\{h^{p,q}\}_{p+q=n}$ be a collection of integers satisfying

$$h^{p,q} = h^{q,p}, \qquad \sum_{p+q=n} h^{p,q} = \operatorname{rank} V.$$

The *period domain* associated to $(V, H, \{h^{p,q}\})$, denoted $D = D(V, H, \{h^{p,q}\})$, is the subset of the flag variety

$$\mathbf{Flag}(V \otimes \mathbf{C}, \{h^{p,q}\}) = \{(F^p \subseteq V \otimes \mathbf{C})_{p \in \mathbf{Z}} \,|\, F^p \subseteq F^{p-1}, \dim(F^p/F^{p+1}) = h^{p,n-p}\}$$

defined by the following conditions

(i) $V \otimes \mathbf{C} = F^p \oplus \overline{F^{q+1}}$, so that $V \otimes \mathbf{C} = \bigoplus_{p+q=n} V^{p,q}$, $V^{p,q} = F^p \cap \overline{F^q}$,
(ii) $H(V^{p,q}, V^{p',q'}) = 0$ for $p \neq p'$,
(iii) $(-1)^{n(n-1)/2+q} H(\alpha,\alpha) > 0$ for $0 \neq \alpha \in V^{p,q}$.

Note that condition (ii) can be rephrased as $F^p = (F^{n-p+1})^\perp$ with respect to the bilinear form $Q(\alpha, \beta) = H(\alpha, \overline{\beta})$, and hence defines a projective subvariety of the flag variety F. Conditions (i) and (iii) are open. Thus D is a complex analytic space. In fact, condition (ii) defines a smooth subvariety, so D is a complex manifold.

In fact, the period domain D is a homogeneous space ([1, §4.4]) for the unitary group U associated to (V, H). To see this, we fix an element $x = (F^\bullet) \in D$ and show that there exists an *adapted* basis (v_i) of V: an orthonormal basis preserved under conjugation and such that each $V^{p,q}$ is spanned by a subset of this basis. If now $y = (G^\bullet) \in D$ is another element, with adapted basis (v_i'), then the element of U sending v_i to v_i' sends x to y; thus the action of U is transitive.

Since the pairing H is not definite, U may not be compact; in fact, it is isomorphic to $\mathrm{Sp}(\frac{1}{2}\dim V, \mathbf{R})$ for n odd and $\mathrm{SO}(a, b)$ where a (resp. b) is the sum of $h^{p,q}$ with p even (resp. odd) for n even. However, the stabilizer K of a point x can be identified with the product of the unitary groups $U(V^{p,q}, H)$ with $p < n/2$ and possibly the special orthogonal group $\mathrm{SO}(V^{n/2,n/2}, H)$ if n is even. Since H is definite on each $V^{p,q}$, K is compact.

Example 2.8. We continue our example with elliptic curves. The flag variety F is simply $\mathbf{P}^1(\mathbf{C})$, condition (i) means removing $\mathbf{P}^1(\mathbf{R})$, condition (ii) is empty, and condition (iii) means that the imaginary part is positive. Thus $D \subseteq \mathbf{P}^1(\mathbf{C}) \setminus \mathbf{P}^1(\mathbf{R})$ is the upper-half plane. The group $U \simeq \mathrm{Sp}(2, \mathbf{R}) \simeq SU(1,1)$ and the stabilizer of $i \in D$ is $K \simeq U(1)$.

3. Variations of Hodge structures

3.1. **Definitions.** Let $f\colon X \to S$ be a smooth and projective morphism to a smooth complex variety S. Then the polarized integral Hodge structures on the cohomology groups $H^n(X_s, \mathbf{Q})$ yield a *polarized variation of Hodge structures* on the associated local system $\mathscr{H} = R^n f_* \mathbf{Q}$. The definition uses the Hodge filtration in place of the Hodge decomposition since the former behaves holomorphically in families and the latter does not.

Definition 3.2. Let S be a complex manifold and let n be an integer.

(1) An *integral variation of Hodge structures of weight* n on S is a local system $\mathscr{H}$ of free finite rank $\mathbf{Z}$-modules on S endowed with a separated and exhaustive descending filtration F^p of the associated holomorphic bundle $\mathscr{H} \otimes \mathscr{O}_S$ by holomorphic subbundles which are locally direct summands satisfying
$$\mathscr{H}_s = F_s^p \oplus \overline{F_s^{q+1}} \text{ for each } s \in S \text{ and } p + q = n$$
and satisfying the *Griffiths transversality* condition
$$\nabla(F^p) \subseteq F^{p-1} \otimes \Omega_S^1$$
where $\nabla\colon \mathscr{H} \otimes \mathscr{O}_S \to (\mathscr{H} \otimes \mathscr{O}_S) \otimes \Omega_S^1$ is the canonical connection.

(2) A *polarized integral variation of Hodge structures of weight* n on S is an integral variation of Hodge structures $\mathscr{H}$ of weight n on S endowed with a bilinear pairing
$$Q\colon \mathscr{H} \times \mathscr{H} \to \mathbf{Z}$$
such that for every $s \in S$, Q makes the Hodge structure $\mathscr{H}_s$ into a polarized Hodge structure of weight n.

(3) An integral variation of Hodge structures of weight n is *polarizable* if locally on S it admits a polarization as above.

The local system $\mathscr{H} = (R^n f_* \mathbf{Z})/\text{tors.}$ attached to a smooth and projective $f\colon X \to S$ underlies a polarizable integral variation of Hodge structures. The filtration F^p is defined by

$$F^p = \operatorname{im}\left(R^n f_* \Omega^{\bullet \geq p}_{X/S} \to R^n f_* \Omega^\bullet_{X/S}\right) \subseteq R^n f_* \Omega^\bullet_{X/S} = R^n f_* \mathbf{C},$$

and the form Q is the intersection form induced by a relatively ample class L.

The Griffiths transversality condition follows from an explicit construction of the Gauss–Manin connection. We will now prove it in the case when $\dim S = 1$ (which is enough because one can detect Griffiths transversality by restricting to curves). In this situation, one has a short exact sequence of complexes[2]

$$0 \longrightarrow \Omega^{\bullet - 1}_{X/S} \otimes f^* \Omega^1_S \longrightarrow \Omega^\bullet_X \longrightarrow \Omega^\bullet_{X/S} \longrightarrow 0. \tag{3.2.1}$$

Applying $R^n f_*(-)$, one obtains the boundary map

$$\delta\colon R^n f_* \Omega^\bullet_{X/S} \longrightarrow R^{n+1} f_* (\Omega^{\bullet - 1}_{X/S} \otimes f^* \Omega^1_S) = R^n f_* \Omega^\bullet_{X/S} \otimes \Omega^1_S,$$

which was shown by Katz and Oda [5] to coincide with the Gauss–Manin connection. Replacing now $\Omega^\bullet_{X/S}$ with $\Omega^{\bullet \geq p}_{X/S}$, one has the short exact sequence mapping to (3.2.1)

$$0 \longrightarrow \Omega^{\bullet - 1 \geq p-1}_{X/S} \otimes f^* \Omega^1_S \longrightarrow \Omega^{\bullet \geq p}_X \longrightarrow \Omega^{\bullet \geq p}_{X/S} \longrightarrow 0,$$

and the two boundary maps fit inside a commutative square

$$\begin{array}{ccc} R^n f_* \Omega^{\bullet \geq p}_{X/S} & \xrightarrow{\delta} & R^n f_* \Omega^{\bullet \geq p-1}_{X/S} \otimes \Omega^1_S \\ \downarrow & & \downarrow \\ R^n f_* \Omega^\bullet_{X/S} & \xrightarrow[\delta = \nabla]{} & R^n f_* \Omega^\bullet_{X/S} \otimes \Omega^1_S. \end{array}$$

Since the images of the vertical maps are by definition F^p and $F^{p-1} \otimes \Omega^1_S$, we deduce that $\nabla(F^p) \subseteq F^{p-1} \otimes \Omega^1_S$.

3.3. **Period maps.** Let $\mathscr{H}$ be a polarized integral variation of Hodge structures on a connected pointed $(S, 0)$, and let $\pi\colon (\tilde{S}, 0) \to (S, 0)$ be the universal covering. The pull-back $\pi^* \mathscr{H}$ is a variation on $\tilde{S}$ whose underlying local system is canonically isomorphic to the constant system with value $\mathscr{H}_0$, compatibly with the pairing. One can therefore view $\pi^* \mathscr{H}$ as a family of Hodge filtrations on $\mathscr{H}_0$ parametrized by points of $\tilde{S}$. There is a unique holomorphic map

$$f\colon \tilde{S} \to D = D(\mathscr{H}_0, Q, \{h^{p,q}\}) = U/K, \quad K = \text{stabilizer of } f(0)$$

such that if $\tilde{s} \in \tilde{S}$ and $f(\tilde{s}) = (F^\bullet) \in D$, then $F^p \mathscr{H}_s = F^p$. We call f the *period mapping* associated to $\mathscr{H}$.

The local system underlying $\mathscr{H}$ corresponds to a homomorphism $\sigma\colon \pi_1(S, 0) \to U \subseteq \mathrm{GL}(\mathscr{H}_0)$, and the map f is equivariant with respect to the natural action of the groups on the source and target of f. The period map thus descends to a map

[2]For $\dim S > 1$, one has a filtered complex instead, and below one would have to deal with spectral sequences rather than cohomology exact sequences.

$S = \tilde{S}/\pi_1(S,0) \to D/\Gamma$ where $\Gamma \subseteq U$ is the image of σ, fitting inside a commutative square

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{f} & D \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ S & \longrightarrow & D/\Gamma. \end{array}$$

The Griffiths transversality condition ensures that the map f is *horizontal*, which means that the differential

$$df\colon T_s S \longrightarrow T_{f(s)} D \subseteq \bigoplus_p \operatorname{Hom}(F^p, V/F^p)$$

has image in $\bigoplus \operatorname{Hom}(F^p/F^{p+1}, F^{p-1}/F^p)$.

3.4. **Semisimplicity and direct summands.** We will need the following fundamental result of Deligne:

Theorem 3.5 (Deligne [3, 4.2.6], see also [2, 1.11, 1.12])**.** *Let $(S,0)$ be a compactifiable (cf. Theorem 2(b)) pointed connected complex manifold, and let $\mathscr{H}$ be a local system underlying an integral polarizable variation of Hodge structures. Then $\mathscr{H}$ is semisimple.*

To prove Theorem 2(b), we will need to show that a direct summand of a local system underlying a polarized variation of Hodge structures does itself admit a polarized variation of Hodge structures.

Proposition 3.6 (cf. [2, 1.13])**.** *Suppose that S is compactifiable, and let $\mathscr{H}$ be a local system underlying a polarizable variation of Hodge structures on S. Let $\mathscr{V}$ be a direct summand of $\mathscr{H}$. Then $\mathscr{V}$ admits a polarized variation of Hodge structures.*

Proof (sketch). By a result of Schmid (using compactifiability) related to the Theorem of the Fixed Part, the components of a horizontal section of $\mathscr{H} = \bigoplus \mathscr{H}^{p,q}$ are horizontal. Applying this not to $\mathscr{H}$ but to $\mathscr{E}nd(\mathscr{H})$, we obtain a grading on global sections

$$\operatorname{End}(\mathscr{H}) = \bigoplus_{p+q=n} \operatorname{End}(\mathscr{H})^{p,q},$$

compatible with the algebra structure.

On the other hand, again by compactifiability, $\mathscr{H}$ is semisimple, which means that $\mathscr{H}$ decomposes

$$\mathscr{H} = \bigoplus_{i \in I} \mathscr{V}_i \otimes W_i$$

where $\mathscr{V}_i$ are irreducible local systems and W_i are complex vector spaces. Consequently, by Schur's lemma we have

$$\operatorname{End}(\mathscr{H}) = \prod_{i \in I} \operatorname{End}(W_i).$$

Our $\mathscr{V}$ is isomorphic to a direct sum of some copies of the $\mathscr{V}_i$, and hence we can suppose that $\mathscr{V} = \mathscr{V}_{i_0}$ for some i_0.

A purely algebraic lemma shows that every grading of $\prod \operatorname{End}(W_i)$ compatible with the algebra structure comes from gradings of the W_i. If we fix such gradings, and pick a homogeneous line $L \subseteq W_{i_0}$, then there exists a homogenous idempotent of degree zero in $\mathrm{End}(W_i)$ whose image in L. This idempotent defines a homogeneous projection of $\mathscr{H}$ onto $\mathscr{V}_i \otimes L \cong \mathscr{V}_i$. Thus $\mathscr{V}_i$ underlies a polarized variation of Hodge structures which is a sub-variation of $\mathscr{H}$ up to renumbering. □

4. Proof of Theorems 1 and 2

4.1. **Griffiths' version of the Schwarz lemma.** The classical Schwarz lemma in complex analysis states that every holomorphic function

$$f\colon \Delta \to \Delta, \quad \Delta = \{|z| < 1\} \subseteq \mathbf{C}, \quad f(0) = 0$$

satisfies $|f'(0)| \leq 1$, with equality if and only if f is a rotation; the proof is an easy application of the maximum principle to $f(z)/z$.

A more appealing equivalent formulation of this lemma says that if we endow Δ with its Poincaré metric $d(x, y)$ (constant curvature -1), then every holomorphic $f\colon \Delta \to \Delta$ (not necessarily preserving the origin) satisfies

$$d(f(x), f(y)) \leq d(x, y), \quad x, y \in \Delta$$

with equality if and only if $x = y$ or f an isometry (a fractional linear transformation). This formulation is the beginning of the long story of the interplay between complex and hyperbolic geometry, Kobayashi manifolds etc.

The following fundamental result of Griffiths is a version of the Schwarz lemma where the target of the map is a period domain. It says that period domains tend to be 'hyperbolic.'

Theorem 4.2 ([4, 10.1])**.** *Let $D = D(\{h^{p,q}\}) = U/K$ be a period domain as in §2.6.There exists a U-invariant Riemannian metric $d_D(x, y)$ on D such that for every horizontal holomorphic map*

$$f\colon \Delta \longrightarrow D$$

one has

$$d_D(f(x), f(y)) \leq d(x, y), \quad x, y \in \Delta.$$

Here $d(x, y)$ is again the Poincaré metric on Δ.

Here a map $f\colon S \to D$ is *horizontal* if

$$df\colon T_s S \longrightarrow T_{f(s)} D \subseteq \bigoplus_p \mathrm{Hom}(F^p, V/F^p)$$

has image in $\bigoplus \mathrm{Hom}(F^p/F^{p+1}, F^{p-1}/F^p)$. The Griffiths transversality condition implies that the period map associated to a polarized variation of Hodge structures is horizontal.

Example 4.3. Coming back to the elliptic curve example: $D = U/K$ is the upper half-plane, which is itself biholomorphic to the unit disc Δ. Thus the assertion of Griffiths' result is the usual Schwarz lemma.

4.4. **Bound on traces.** The fundamental input, based on Griffiths' theorem above, is a bound on the trace of a fixed element $\gamma \in \pi_1(S, 0)$.

Proposition 4.5. *Let $(S, 0)$ be a connected pointed complex manifold, and let $\gamma \in \pi_1(S, 0)$. Then for every $N \geq 0$ there exists a constant $C > 0$ such that for every polarized variation of Hodge structures $\mathscr{H}$ of rank N on S, the corresponding representation $\sigma\colon \pi_1(S, 0) \to \mathrm{GL}(\mathscr{H}_0)$ satisfies*

$$|\mathrm{Tr}(\sigma(\gamma))| < C.$$

Proof. We fix a basis $\{v_i\}$ of $\mathscr{H}_0 \otimes \mathbf{C}$ which is *adapted* in the sense that each F^p is spanned by a subset of this basis and such that $H(v_i, v_j) = \delta_{ij}$.

Let $\pi\colon (\tilde{S}, 0) \to (S, 0)$ be the universal covering. Then the pullback $\pi^*\mathscr{H}$ is a polarized variation of Hodge structures on $\tilde{S}$ whose underlying local system is constant with value $\mathscr{H}_0$. Let $D = D(\mathscr{H}_0, H, \{h^{p,q}\}) = U/K$ be the associated period domain and let d_D be the U-invariant metric in Theorem 4.2. We have the period map

$$f\colon \tilde{S} \longrightarrow D,$$

which is equivariant with respect to the map $\sigma\colon \pi_1(S, 0) \to U \subseteq \mathrm{GL}(\mathscr{H}_0)$.

Since $\tilde{S}$ is connected, there exist points

$$0 = x_0, x_1, \ldots, x_r = \gamma \cdot 0,$$

holomorphic maps

$$u_1, \ldots, u_r\colon \Delta \longrightarrow \tilde{S},$$

and points $a_1, \ldots, a_{r-1}, b_1, \ldots, b_r \in \Delta$ such that

$$x_0 = u_1(a_1), \quad u_1(b_1) = x_1 = u_2(a_2), \quad \ldots, \quad u_r(b_r) = x_r.$$

Applying Theorem 4.2 to the compositions $f_i = f \circ u_i\colon \Delta \to D$, we obtain an upper bound

$$d_D(f(0), f(\gamma \cdot 0)) \leq \sum_{i=1}^{r} d_D(f(x_{i-1}), f(x_i))) = \sum_{i=1}^{r} d_D(f_i(a_i), f_i(b_i)) \leq \sum_{i=1}^{r} d(a_i, b_i).$$

(See Figure 1.) This means that $f(\gamma \cdot 0) = \sigma(\gamma) \cdot 0$ lies in a bounded subset of $D = U/K$ depending only on $\{h^{p,q}\}$. Since K is compact and d_D is U-invariant, this means that $\sigma(\gamma) \in U \subseteq \mathrm{GL}(\mathscr{H}_0)$ lies in a bounded subset of U. Consequently, the matrix coefficients of $\sigma(\gamma)$ in the basis $\{v_i\}$ are bounded, and therefore so is its trace.

The bound we have obtained depends on $\{h^{p,q}\}$; in the last step of the proof, we have to deal with the fact that the weight n of $\mathscr{H}$ can be unbounded.

First, we observe that if $h^{p,q} = 0$ for some p and q, then F^{p+1} is horizontal. Indeed, Griffiths transversality asserts that

$$\nabla(F^{p+1}) \subseteq F^p \otimes \Omega^1_S,$$

but $F^p/F^{p+1} = H^{p,q} = 0$. Consequently, the polarized variation $\mathscr{H}$ decomposes into a direct sum of polarized variations for which the set $\{p \mid h^{p,n-p} \neq 0\}$ is a connected interval.

Second, if $\{p \mid h^{p,n-p} \neq 0\}$ is a connected interval, then after suitable renumbering and changing H by a sign, we can assume that this interval starts at 0 and that $n \leq N$. Now there are finitely many possibilities for the $h^{p,q}$. □

4.6. **Finite determination of traces.** We need a purely group-theoretic result:

Proposition 4.7. *Let Γ be a finitely group and $N \geq 0$ an integer. There exists a finite subset $F \subseteq \Gamma$ such that if $\sigma_i\colon \Gamma \to \mathrm{GL}_N(\mathbf{C})$ ($i = 1, 2$) are two representations such that* $\mathrm{Tr}(\sigma_1(\gamma)) = \mathrm{Tr}(\sigma_2(\gamma))$ *for all $\gamma \in F$, then one has* $\mathrm{Tr}(\sigma_1(\gamma)) = \mathrm{Tr}(\sigma_2(\gamma))$ *for all $\gamma \in \Gamma$.*

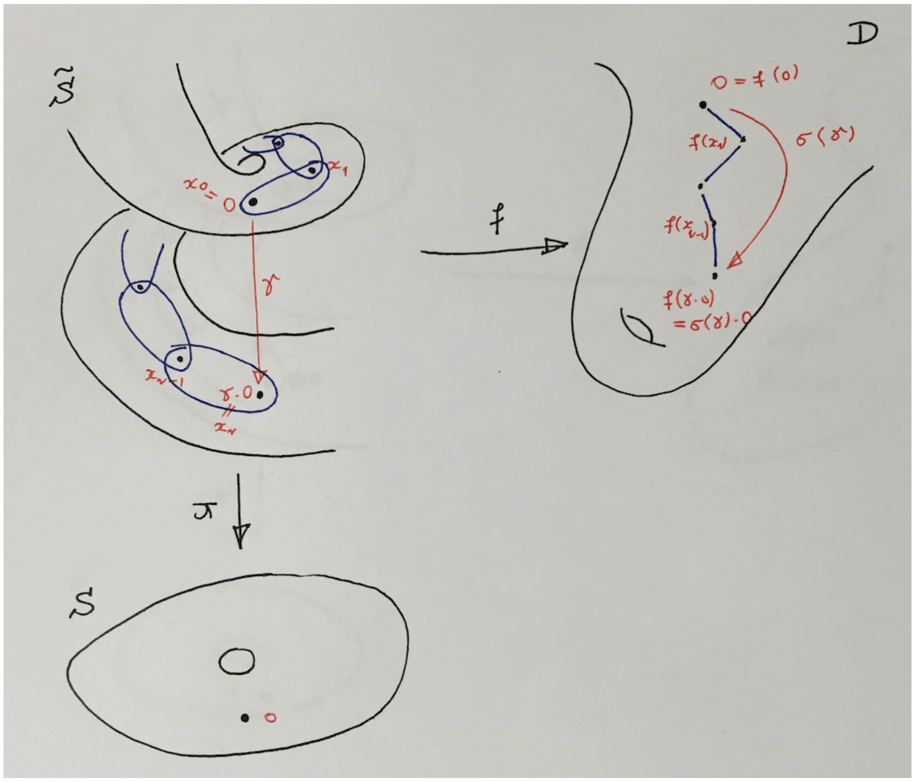

FIGURE 1. Proof of Proposition 4.5

Proof. Clearly we can assume that Γ is the free monoid on a set T. Consider the free algebra A over $\mathbf{C}$ with indeterminates x_{ij}^t for $i, j = 1, \dots, N$ and $t \in T$; we have a natural representation $\sigma \colon \Gamma \to M_{N \times N}(A)$ sending t to $[x_{ij}^t]$, and an action of $\mathrm{GL}_N(\mathbf{C})$ on A by conjugation. For $\gamma = t_1 \cdot \ldots \cdot t_r \in \Gamma$, $t_i \in T$, the trace

$$\mathrm{Tr}(\sigma(\gamma)) \in A$$

lies in the subring of invariants A^{GL_N}.

Since GL_N is reductive, this subring is finitely generated. Moreover, Procesi [7] has shown that it is in fact generated by the elements $\mathrm{Tr}(\sigma(\gamma))$, $\gamma \in \Gamma$. Thus there exists a finite set $F \subseteq \Gamma$ such that $\mathrm{Tr}(\sigma(\gamma))$ $(\gamma \in F)$ generate A^{GL_N}, and consequently for every $\gamma \in \Gamma$, the element $\mathrm{Tr}(\sigma(\gamma))$ is a polynomial in $\mathrm{Tr}(\sigma(\gamma'))$ for $\gamma' \in F$.

Specializing the x_{ij}^t to the matrix coefficients of $\sigma_i(t)$, we see that $\mathrm{Tr}(\sigma_i(\gamma))$ is a polynomial (independent of σ) of $\mathrm{Tr}(\sigma_i(\gamma'))$ for $\gamma' \in F$. □

4.8. **Proof of Theorem 2.**

Proof. (a) Let $\mathscr{H}$ be a polarized integral variation of Hodge structures of rank N on S, and let $\sigma \colon \pi_1(S, 0) \to \mathrm{GL}(\mathscr{H}_0)$ be the associated representation. Then $\mathrm{Tr}(\sigma(\gamma)) \in \mathbf{Z}$ for every $\gamma \in \pi_1(S, 0)$, and $|\mathrm{Tr}(\sigma(\gamma))| < C$ where C depends only on S and N, by Proposition 4.5. Therefore $\mathrm{Tr}(\sigma(\gamma))$ can take only finitely many possible values.

Let $F \subseteq \pi_1(S,0)$ be a finite set as in Proposition 4.5. Then the restriction of $\mathrm{Tr}(\sigma(\gamma))$ to F can take only finitely many values. Consequently, by the property of the set F, there are only finitely many possibilities for $\gamma \mapsto \mathrm{Tr}(\sigma(\gamma))$.

(b) Let $\mathscr{V}$ be a subquotient of a local system $\mathscr{H}$ underlying a polarized integral variation of Hodge structures. By §3.4, $\mathscr{V}$ underlies a polarized variation of Hodge structures, and hence by the proof of (a) there are only finitely many possible $\mathscr{V}$ up to semisimplification. By Theorem 3.5, there are only finitely many possible $\mathscr{V}$ up to isomorphism. □

4.9. **Proof of Theorem 1.**

Proof. By Nagata and Hironaka, S^{an} is compactifiable, so we can apply Theorem 2(b). Suppose that $\mathscr{V}$ is a local system on S^{an} which comes from geometry; by definition, there exists a smooth projective $f\colon X \to S$ such that $\mathscr{V}$ is a direct summand of $\mathscr{H} = R^n f_* \mathbf{Q}$. The local system $\mathscr{H}$ underlies an integral polarized variation of Hodge structure, and hence by Theorem 2(b) $\mathscr{V}$ belongs to a finite set of isomorphism classes. □

5. ℓ-adic version

Let k be a finitely generated field with algebraic closure $\overline{k}$ and let X be a geometrically connected, normal and quasi-projective scheme over k. Let $\overline{x} \in X(\overline{k})$. One has the exact sequence

$$1 \to \pi_1(X_{\overline{k}}, \overline{x}) \to \pi_1(X, \overline{x}) \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

Here if k has positive characteristic and X is not complete, we denote by π_1 the *tame* fundamental group.

Let ℓ be a prime invertible in k and let L be a finite extension of $\mathbf{Q}_\ell$ with the natural topology. Let us call a continuous representation

$$\sigma\colon \pi_1(X_{\overline{k}}, \overline{x}) \longrightarrow \mathrm{GL}_N(L)$$

arithmetic if there exists a finite extension k' of k and a continuous representation $\rho\colon \pi_1(X_{k'}, \overline{x}) \to \mathrm{GL}_M(L)$ such that σ is isomorphic to a subquotient of the restriction of ρ to $\pi_1(X_{\overline{k}}, \overline{x})$.

Theorem 5.1 (Litt[3] [6]). *For X as above and $N \geq 0$, there exist only finitely many isomorphism classes of semisimple arithmetic continuous representations*

$$\sigma\colon \pi_1(X_{\overline{k}}, \overline{x}) \longrightarrow \mathrm{GL}_N(L).$$

Let now k be any field with algebraic closure k, and let X/k be as before. We say that a continuous representation

$$\sigma\colon \pi_1(X_{\overline{k}}, \overline{x}) \longrightarrow \mathrm{GL}_N(L)$$

comes from geometry [6, 3.1.6] if it is a subquotient of the representation attached to $R^i f_* L$ for some smooth and proper $f\colon Y \to X$, or more generally such an f defined over some algebraically closed extension of k. Deligne has shown that such representations are semisimple. Moreover, if k is finitely generated, then σ is arithmetic. Since every variety can be defined over a finitely generated field by spreading out, one can deduce the following:

[3]In fact in the paper this is stated for curves, but the general case can be deduced by a Lefschetz-type argument, see [6, Remark 1.1.9]. The formulation given here follows a private communication with the author.

Theorem 5.2 (Litt [6])**.** *Let k be a field and let X be a geometrically integral, normal, and quasi-projective scheme over k. Let $\overline{x} \in X(\overline{k})$. Let L be a finite extension of $\mathbf{Q}_\ell$ where ℓ is a prime invertible in k. Then there exist only finitely many continuous representations*

$$\pi_1(X_{\overline{k}}, \overline{x}) \longrightarrow GL_N(L)$$

which come from geometry.

Thus, in a sense, the extension of a representation of $\pi_1(X_{\overline{k}}, \overline{x})$ to the arithmetic fundamental group $\pi_1(X, \overline{x})$ (up to passing to a finite extension of k) is the arithmetic analog of finding a polarized variation of Hodge structures on a given rational local system on a smooth complex variety.

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