INTRO TO DEFORMATION THEORY (MAR 3, 2016)

PIOTR ACHINGER

This is a note based on my talk on March 3. I discuss the so-called Yoneda (or 'functor of points') philosophy in algebraic geometry, and some basics of deformation theory (without Schlessinger's criteria, deformations of schemes, and obstruction classes – these will hopefully appear later in the seminar). There is plenty of good introductory texts to deformation theory, see e.g. [FG05] (the first chapter) or the very readable book [Har10].

1. The functor of points approach

We begin with a somewhat philosophical discussion of the Yoneda lemma in category theory. Let C be category, x an object of C. Consider the contravariant functor $h_x : C^{\rm op} \to (\text{Sets})$ taking an object y to the set $\text{Hom}_C(y, x)$, and a morphism $f : y \to y'$ to the map $\text{Hom}_C(y', x) \to \text{Hom}_C(y, x)$ given by precomposition with f. Varying x, we obtain a (covariant!) functor $C \to \text{Fun}(C^{\rm op}, (\text{Sets}))$ to the category $\text{Fun}(C^{\rm op}, (\text{Sets}))$ of functors $C^{\rm op} \to (\text{Sets})$ (with morphisms given by natural transformations of functors), sending x to h_x and a morphism $f : x \to x'$ to the natural transformation $\text{Hom}_C(y, x) \to \text{Hom}_C(y, x')$ given by postcomposition with f.

Lemma 1.1 (Yoneda)**.** *The functor $x \mapsto h_x : C \to \text{Fun}(C^{\rm op}, (\text{Sets}))$ is fully faithful. More generally, for any functor $F : C^{\rm op} \to (\text{Sets})$ and any object x of C, the map associating to an element $\xi \in F(x)$ the system of maps $\text{Hom}(y, x) \to F(y)$ sending $f : y \to x$ to $F(f)(\xi) \in F(y)$ gives a bijection $F(x) \to \text{Hom}(h_x, F)$.*

In other words, natural transformations $h_x \to h_{x'}$ are in bijection with maps $x \to x'$ in C, and thus C can be treated as a full subcategory of $\text{Fun}(C^{\rm op}, (\text{Sets}))$. The second assertion states that for a functor $F : C^{\rm op} \to (\text{Sets})$, the restriction of the functor $h_F : \text{Fun}(C^{\rm op}, (\text{Sets}))^{\rm op} \to (\text{Sets})$ to $C^{\rm op} \subseteq \text{Fun}(C^{\rm op}, (\text{Sets}))^{\rm op}$ is canonically isomorphic to F.

For us, the main point of the Yoneda lemma is as follows. Suppose that $F : C^{\rm op} \to (\text{Sets})$ is any functor. By Yoneda, it makes sense to ask whether F *is* an object of C, that is, whether there exists an object x of C and an isomorphism $\iota : h_x \to F$ (if so, we call F *representable*, and x an object *representing* F). Indeed, if $h_x \simeq F \simeq h_{x'}$, then $x \simeq x'$ by the first assertion of the lemma, so x is well-defined. Moreover, the second assertion implies that the choice of ι is equivalent to giving an element $\xi \in F(x)$. The advantage of this point of view is that *it is often easier to define a functor $F : C^{\rm op} \to (\text{Sets})$ than an object of C.*

This is perhaps still too abstract, so let us give a few case studies of applications of this philosophy in algebraic geometry. As we will see, sometimes it is impossible to avoid using this paradigm even when dealing with very concrete questions.

1.2. **Products.** Given two $\mathbb{C}$-schemes X and Y, we would like to define $X \times Y$. As a set, this should be the cartesian product of X and Y, and it is easy to construct $X \times Y$ as a scheme if X and Y are both affine (by taking tensor product of the corresponding algebras). The construction in the general case involves a non-canonical choice of affine open coverings of X and Y and gluing. Is is then proved that for any $\mathbb{C}$-scheme Z, there is a natural bijection $\mathrm{Hom}(Z, X \times Y) \simeq \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y)$, given by composition with the two projections $X \times Y \to X$, $X \times Y \to Y$.

Let C be the category of $\mathbb{C}$-schemes, and let $F : C^{\mathrm{op}} \to (\mathrm{Sets})$ be the functor

$$F(Z) = \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y).$$

(So $F = h_X \times h_Y$)). The property of $X \times Y$ mentioned above implies that we have $h_{X\times Y} \simeq F$, so by Yoneda $X \times Y$ is uniquely determined by F. So instead of constructing $X \times Y$ in an ad-hoc way, we could have started with $F = h_X \times h_Y$ and asked whether it is representable. The construction of $X \times Y$ by patching yields a proof that F is representable. This discussion applies more generally to products (or more generally – limits) in an arbitrary category.

1.3. **Projective spaces.** Suppose a newcomer to algebraic geometry wanted to define the projective space $\mathbb{P}(V)$ of a finite dimensional complex vector space V. As a set, this should consist of one-dimensional linear subspaces of V, and the problem is to give this set a natural structure of an algebraic variety. Classically, this is done either by gluing together a number of affine spaces $\mathbb{A}^n$ (which requires choosing a basis of V), or a bit more canonically, using the Proj construction. What is perhaps lacking here is a clear explanation why this is *the* correct structure. To add confusion, a line bundle $\mathscr{O}(1)$ is introduced by a rather ad-hoc procedure. Later on, when studying about line bundles and linear series, one learns that globally generated line bundles yield maps to projective spaces. If $f : X \to \mathbb{P}(V)$ is a map, we get the line bundle $L = f^*\mathscr{O}(1)$, together with a map $p : V^* = H^0(\mathbb{P}(V), \mathscr{O}(1)) \to H^0(X, L)$ with the property that for every point $x \in X$, there exists a $\varphi \in V^*$ such that $p(\varphi)$ does not vanish at x. In other words, the associated map $V^* \otimes \mathscr{O}_X \to L$ is a surjective map of sheaves on X. It is easy to see that the association $f \mapsto (L, p)$ is a bijection.

Proposition 1.4. *For any scheme X over $\mathbb{C}$, the above construction gives a bijection*

$$\mathrm{Hom}(X, \mathbb{P}(V)) \to \{\textit{line bundles } L \textit{ on } X \textit{ together with a surjection } V^* \otimes \mathscr{O}_X \to L\}.$$

Note that by duality, surjections $V^* \otimes \mathscr{O}_X \to L$ correspond to maps $L \to V \otimes \mathscr{O}_X$ such that for every $x \in X$, the map $L \otimes k(x) \to V \otimes \mathscr{O}_X \otimes k(x) = V \otimes k(x)$ is injective (we call this a *line subbundle* of $V \otimes \mathscr{O}_X$). Thus a map $f : X \to \mathbb{P}(V)$ yields a family of linear subspaces L of V parameterized by the $\mathbb{C}$-points of X, which is a very natural generalization of the most naive definition of $\mathbb{P}(V)$ as the *set* of all lines in V.

What does Yoneda tell us about this situation? Let C be again the category of $\mathbb{C}$-schemes, and let $F : C^{\mathrm{op}} \to (\mathrm{Sets})$ be the functor

$$\begin{aligned} F_V(X) &= \{\text{line subbundles of } V \otimes \mathscr{O}_X\} \\ &= \{\text{line bundles } L \text{ on } X \text{ together with a surjection } p : V^* \otimes \mathscr{O}_X \to L\}. \end{aligned}$$

Then Proposition 1.4 asserts that we have a natural isomorphism $\iota : h_{\mathbb{P}(V)} \simeq F_V$. By Yoneda, this shows that F_V determines $\mathbb{P}(V)$ uniquely. Moreover, the second assertion of Yoneda shows that ι corresponds to an element $\xi \in F_V(\mathbb{P}(V))$, that is, a surjection $V^* \otimes \mathscr{O}_{\mathbb{P}(V)} \to L$ onto a line bundle L. This L is just $\mathscr{O}(1)$.

This gives the following way of defining $\mathbb{P}(V)$ and $\mathscr{O}(1)$. We start with the functor F_V above, and define $\mathbb{P}(V)$ to be any scheme representing F_V, i.e., a scheme P together with an element $\xi = (L, p) \in F_V(P)$ such that the associated (by Yoneda) $\iota : h_P \to F_V$ is an isomorphism. Then $\mathscr{O}(1)$ is simply the line bundle L, and it comes with a natural surjection $p : V \otimes \mathscr{O}_{\mathbb{P}(V)} \to \mathscr{O}(1)$. Of course, one still has to prove that F is representable, but the advantage is that the definition is simple and natural, and all noncanonical choices are pushed into the proof of representability.

One truly starts to appreciate this approach when trying to define more general spaces. For example, it is now almost obvious how correctly to define the Grassmannian $G(k, V)$ of k-dimensional subspaces of V, or the full flag variety $\mathrm{Fl}(V)$ (setting aside the proofs of representability of the corresponding functors). Moreover, it is clear from our definition of $\mathbb{P}(V)$ that it is functorial with respect to injective maps $V \to V'$: we trivially get a natural transformation $F_V \to F_{V'}$, and hence a map $\mathbb{P}(V) \to \mathbb{P}(V')$ by Yoneda. Finally, the generalization of $\mathbb{P}(V)$ to the relative case becomes almost obvious: if S is a scheme and $\mathcal{E}$ a vector bundle on S, we can simply define $\mathbb{P}(\mathcal{E})$ as an S-scheme representing the functor $F_{\mathcal{E}}(X \to S) = \{\text{line subbundles of } f^*\mathcal{E}\}$. Again by Yoneda, it comes with a natural line bundle, the relative $\mathscr{O}(1)$.

1.5. **Hilbert schemes.** To a closed subscheme $Z \subseteq \mathbb{P}^n$ one can associate its Hilbert polynomial $\chi_Z \in \mathbb{Q}[t]$, which is the unique polynomial satisfying $\chi_Z(k) = \dim \Gamma(Z, \mathscr{O}_Z(k))$ for $k \gg 0$. If S is connected and $\mathcal{Z} \subseteq \mathbb{P}^n \times S$ is a *flat* family of closed subschemes of $\mathbb{P}^n$, then the Hilbert polynomial χ_{Z_s} of $Z_s = \mathcal{Z} \cap \mathbb{P}^n \times \{s\}$ is independent of s.

Let $P \in \mathbb{Q}[t]$ be a polynomial taking integer values. The *Hilbert scheme* Hilb^n_P is a scheme parameterizing (i.e., whose points are in bijection with) closed subschemes $Z \subseteq \mathbb{P}^n$ with Hilbert polynomial $\chi_Z = P$. Not only is it unclear whether such a thing exists; the previous sentence cannot even serve as its definition. But, given our experience from the previous paragraphs, we eagerly define the functor

$$\begin{aligned} H^n_P(X) = \{&\text{closed subschemes } \mathcal{Z} \subseteq \mathbb{P}^n \times X, \text{ flat over } X, \\ &\text{and such that } \chi_{Z_x} = P \text{ for all } x \in X\}, \end{aligned}$$

and *define* Hilb^n_P to be a scheme representing the functor H^n_P. Now we can ask: is the functor H^n_P representable? The answer turns out to be *yes*, but the proof is not easy. But at least we can ask an honest question, with answer either *yes* or *no*, as opposed to the vague 'does there exist a scheme whose points are in natural bijection with...?' (we could take just a disjoint union of points).

It turns out that $H := \mathrm{Hilb}^n_P$ is proper over $\mathbb{C}$ (i.e. complete, or compact). Recall the valuative criterion of properness [Har77, II §3]: H is proper if and only if it is of finite type and for every valuation ring R and $f : \mathrm{Spec}(\mathrm{Frac}(R)) \to H$, there exists a unique extension $\overline{f} : \mathrm{Spec}(R) \to H$. Suppose that we know H is of finite type. By definition of Hilb^n_P, f corresponds to a closed subscheme $Z \subseteq \mathbb{P}^n_K$ with Hilbert polynomial P. Let $\overline{Z}$ be the closure of Z in $\mathbb{P}^n_R$. Then $\overline{Z}$ is flat over R (REF), and hence $\chi_{\overline{Z}} = \chi_Z = P$. Thus $\overline{Z}$ defines the desired $\overline{f}$, which is moreover unique.

Exercise 1.6. Apply a similar argument to show that $\mathbb{P}(V)$ is complete if it exists and is of finite type, arguing with the functor F_V alone.

This discussion shows that with our approach, we can prove things about Hilb^n_P (properness, granted that it is of finite type) without even knowing whether it exists! A different, much more complicated argument of this sort appeared in Hartshorne's Ph.D. thesis, where he proved that Hilb^n_P is always connected. As he remarks in the introduction, the proof does not use anywhere the difficult fact that H^n_P is representable.

1.7. **The functor of points approach.** Taking this Yoneda-based aproach a little bit further, given a scheme X, we can consider the contravariant functor

$$\underline{X} : (\mathbb{C} - \text{algebras}) \to (\text{Sets}), \quad \underline{X}(R) = \mathrm{Hom}(\mathrm{Spec}\, R, X),$$

called *the functor of points* of X. Using the arrow-reversing equivalence $(\mathbb{C} - \text{algebras}) \simeq (\text{affine } \mathbb{C} - \text{schemes})^{\mathrm{op}}$, we see that the functor $\underline{X}$ coincides with the restriction of $h_X : (\mathbb{C} - \text{schemes})^{\mathrm{op}} \to (\text{Sets})$ to the full subcategory of affine $\mathbb{C}$-schemes.

Proposition 1.8. *The association $X \mapsto \underline{X}$ is fully faithful.*

Proof. Let X and Y be $\mathbb{C}$-schemes, and let $\varphi : \underline{X} \to \underline{Y}$ be a natural transformation. Cover $X = \bigcup U_i$, by open affines. Composing with φ, we get natural transformations $\varphi_i : \underline{U_i} \to \underline{Y}$. By Yoneda applied to the category of affine schemes, the φ_i yield elements f_i of $\underline{(Y)}(U_i) = \mathrm{Hom}(U_i, Y)$. For each i, j, let $U_i \cap U_j = \bigcup U_{ijk}$ be an affine open cover of the double intersection. Then again by Yoneda, the restrictions of f_i and f_j to each U_{ijk} coincide, and thus the f_i glue to a unique morphism $f : X \to Y$. □

In the previous paragraphs, we used Yoneda to identify a scheme X with the corresponding functor $h_X : (\text{Schemes})^{\mathrm{op}} \to (\text{Sets})$. Using the above proposition, we can identify X with its functor of points $\underline{X} : (\mathbb{C} - \text{algebras}) \to (\text{Sets})$. The clear advantage of the latter is that it does not even mention the word *scheme*. So one could define various schemes without even knowing the definition of a scheme, for example

$$\underline{\mathbb{P}}^n(A) = \{\text{direct summands of } A^n \text{ of rank one}\}.$$

We have thus realized the category of schemes as a full subcategory of the category of functors $(\mathbb{C} - \text{algebras}) \to (\text{Sets})$. One can now wonder whether there is a criterion for such a functor to be a scheme. The answer is yes, and the criterion uses the notion of a Zariski sheaf. I discuss this briefly below, feel free to skip this.

Definition 1.9. (1) Let A be a $\mathbb{C}$-algebra. A *Zariski cover* of A is a family $\{A \to A_i\}_{i\in I}$ of $\mathbb{C}$-algebra morphisms, where each $A_i = A[1/f_i]$ for some $f_i \in A$ generating the unit ideal of A.

(2) A functor $F : (\mathbb{C} - \text{algebras}) \to (\text{Sets})$ is a *Zariski sheaf* if for every $\mathbb{C}$-algebra A and for every Zariski cover $\{A \to A_i\}_{i\in I}$, the diagram

$$F(A) \to \prod_{i\in I} F(A_i) \rightrightarrows \prod_{i,j\in I} F(A_{ij}),$$

where $A_{ij} = A_i \otimes_A A_j = A[1/(f_i f_j)]$, is an equalizer. (Recall that a diagram $A \to B \rightrightarrows C$ of sets is called an equalizer if $A \to B$ is injective with image equal to the set of all elements of B whose images in C under the two maps $B \to C$ are equal).

(3) Let $F_i, F : (\mathbb{C} - \text{algebras}) \to (\text{Sets})$ be Zariski sheaves, $\varphi_i : F_i \to F$ natural transformations. We say that the φ_i are *jointly surjective* if for every field K, the map $\bigsqcup F_i(K) \to F(K)$ is surjective.

(4) Let A be a $\mathbb{C}$-algebra, $I \subseteq A$ an ideal. Then $D(I)$ is the following subfunctor of $\mathrm{Hom}(A, -)$:

$$D(I)(B) = \{\varphi : A \to B \,:\, \varphi(I)B = B\}.$$

A morphism $D \to \mathrm{Hom}(A, -)$ of functors $(\mathbb{C} - \text{algebras}) \to (\text{Sets})$ is called an *open subset* if $D \simeq D(I)$ for some ideal I. More generally, for any $F : (\mathbb{C} - \text{algebras}) \to (\text{Sets})$, a subfunctor $D \subseteq F$ is called an open subfunctor if for every $\mathbb{C}$-algebra A and every $\mathrm{Hom}(A, -) \to F$, the preimage of D in $\mathrm{Hom}(A, -)$ is an open subset.

(5) Let $F_i, F : (\mathbb{C} - \text{algebras}) \to (\text{Sets})$ be Zariski sheaves, $\varphi_i : F_i \to F$ natural transformations. We say that the φ_i are *jointly surjective* if for every field K, the map $\bigsqcup F_i(K) \to F(K)$ is surjective. We say that the φ_i are an *open covering* of F is they are jointly surjective and each F_i is an open subfunctor of F.

Proposition 1.10. *A functor $F : (\mathbb{C} - \text{algebras}) \to (\text{Sets})$ is a scheme if and only if F is a Zariski sheaf and there exists $\mathbb{C}$-algebras R_i and $\xi_i \in F(R)$ inducing an open covering $\mathrm{Hom}(R_i, -) \to F$.*

Thus, in principle, one could take the above proposition as the *definition* of a scheme. A clear benefit is that we can avoid using locally ringed spaces, and that some constructions are easier to carry out. For example, it is easy to prove using the above proposition that fiber products of schemes always exist. Of course, to do algebraic geometry on a scheme X, one really needs the locally ringed space, but for some applications (group schemes, for example) the functor of points is more elegant.

2. Infinitesimal geometry

In the previous section we observed that we can safely replace a scheme X with its functor of points $\underline{X} : (\mathbb{C} - \text{algebras}) \to (\text{Sets})$. If $x \in X$ is a $\mathbb{C}$-point, we can study X locally around x by studying the functor

$$\underline{X}_x : (\text{local } \mathbb{C} - \text{algebras with residue field } \mathbb{C}) \to (\text{Sets}),$$

$$A \mapsto \{f : \mathrm{Spec}(A) \to X \ : \ f(\text{closed point}) = x\}.$$

More generally, the above definition makes sense for any $F : (\mathbb{C} - \text{algebras}) \to (\text{Sets})$ and any $x \in F(\mathbb{C})$. In case $F = \underline{X}$ is a scheme, this functor remembers only the local ring $\mathscr{O}_{X,x}$.

Proposition 2.1. *Let $f : \mathrm{Spec}(\mathscr{O}_{X,x}) \to X$ be the canonical map, A a local ring with residue field $\mathbb{C}$, $g : \mathrm{Spec}(A) \to X$ a map sending the closed point to x. Then there exists a unique local homomorphism $\mathscr{O}_{X,x} \to A$ such that the induced $h : \mathrm{Spec}(A) \to \mathrm{Spec}(\mathscr{O}_{X,x})$ satisfies $h \circ f = g$. In particular $\underline{X}_x \simeq \mathrm{Hom}(\mathscr{O}_{X,x}, -)$.*

Proof. Obvious! □

In infinitesimal geometry, one replaces local algebras with Artinian local algebras. Recall that a $\mathbb{C}$-algebra A is Artinian if $\dim_{\mathbb{C}} A = 0$. Let Art denote the category of local Artinian $\mathbb{C}$-algebras. As before, if X is a scheme and $x \in X$ a $\mathbb{C}$-point, we can study the functor

$$\hat{X}_x : \mathrm{Art} \to (\text{Sets}), \quad \hat{X}_x(A) = \{f : \mathrm{Spec}(A) \to X \ : \ f(\text{closed point}) = x\}.$$

This functor only depends on the completion $\hat{\mathscr{O}}_{X,x} = \lim \mathscr{O}_{X,x}/\mathfrak{m}_x^{n+1}$, treated as a topological ring endowed with the inverse limit topology.

Proposition 2.2. *Let $f : \mathrm{Spec}(\hat{\mathscr{O}}_{X,x}) \to X$ be the canonical map, A a an Artinian local $\mathbb{C}$-algebra, $g : \mathrm{Spec}(A) \to X$ a map sending the closed point to x. Then there exists a unique local and continuous homomorphism $\hat{\mathscr{O}}_{X,x} \to A$ such that the induced $h : \mathrm{Spec}(A) \to \mathrm{Spec}(\mathscr{O}_{X,x})$ satisfies $h \circ f = g$. In particular $\underline{X}_x \simeq \mathrm{Hom}_{\mathrm{cont}}(\hat{\mathscr{O}}_{X,x}, -)$.*

Note however that $\hat{\mathscr{O}}_{X,x}$ is usually *not* an object of Art, but only an inverse limit of such. This motivates the definition below.

Definition 2.3. We call a functor $F : \mathrm{Art} \to (\text{Sets})$ *pro-representable* if $F \simeq \mathrm{Hom}(A, -)$ for a complete local $\mathbb{C}$-algebra A such that $A/\mathfrak{m}_A^n$ is in Art for all n.

Later, we will review Schlessinger's criteria for pro-representability, the infinitesimal version of Proposition 1.10.

2.4. **Tangent vectors to the Picard scheme.** One thing we can study using $F = \hat{X}_x$ alone is tangent vectors at x. Recall that a tangent vector at x is a linear map $v : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{C}$. Consider the map $\overline{v} : \hat{\mathscr{O}}_{X,x} \to \mathbb{C}[\varepsilon]/(\varepsilon)$ sending $f \in \mathscr{O}_{X,x}$ to $f(x) + v((f - f(x)) \bmod \mathfrak{m}_x^2) \cdot \varepsilon$. This map is a ring homomorphism, and the association $v \mapsto \overline{v}$ is a bijection. Thus $F(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ is the tangent space to X at x.

To illustrate this, let us compute the tangent space to the Picard variety at 0. Let X be a smooth projective variety, $\mathrm{Pic}(X)$ the set of isomorphism classes of line bundles on X. Tensor product gives $\mathrm{Pic}(X)$ the structure of an abelian group, the Picard group of X. It turns out that $\mathrm{Pic}(X)$ has a natural scheme structure as well. Using the philosophy from the previous section, we can define $\mathrm{Pic}(X)$ as a functor by

$$\underline{\mathrm{Pic}}(X)(A) = \mathrm{Pic}(X \times \mathrm{Spec}(A))/\,\mathrm{Pic}(\mathrm{Spec}\, A).$$

We divide by $\mathrm{Pic}(A)$ because a line bundle on $X \times \mathrm{Spec}(A)$ which is pulled back from $\mathrm{Spec}(A)$ is trivial along all fibers of $X \times \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ and hence does not give an interesting family of line bundles on X parameterized by points of $\mathrm{Spec}(A)$ This functor turns out to be representable, but we don't need to know this to compute its tangent space.

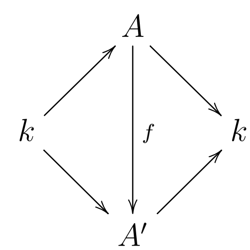
Let $0 \in \mathrm{Pic}(X)$ be the point corresponding to the trivial line bundle $\mathscr{O}_X$. The tangent space to $\mathrm{Pic}(X)$ at 0 is by definition $\hat{\mathrm{Pic}}(X)_0(\mathbb{C}[\varepsilon]/(\varepsilon^2))$, which is the set of isomorphism classes of line bundles on $X[\varepsilon] := X \times \mathrm{Spec}\, \mathbb{C}[\varepsilon]/(\varepsilon^2)$ whose restriction to X is trivial. Let L be such a line bundle, together with an identification $L/\varepsilon L = \mathscr{O}_X$, then L sits inside a short exact sequence

$$0 \to \mathscr{O}_X \xrightarrow{a} L \xrightarrow{b} \mathscr{O}_X \to 0.$$

We treat L as an $\mathscr{O}_X$-module via $\mathscr{O}_X \to \mathscr{O}_X[\varepsilon]$ sending f to $f + 0\varepsilon$. This extension yields a class $v_L \in \mathrm{Ext}^1(\mathscr{O}_X, \mathscr{O}_X) = H^1(X, \mathscr{O}_X)$. On the other hand, given an extension as above, we get a natural action of $\mathscr{O}_X[\varepsilon]$ on L by $\varepsilon = a \circ b$. This gives a bijection between the tangent space to $\mathrm{Pic}(X)$ at 0 and the group $H^1(X, \mathscr{O}_X)$.

3. Functors of Artin rings

Let k be a field. We denote by Art_k the category of Artinian local k-algebras with residue field k, where morphisms $f : A \to A'$ are ring homomorphisms making the diagram



commute. The algebra $k[\varepsilon] := k[\varepsilon]/(\varepsilon^2)$ will be our favorite example of an object of Art_k. It is often called the *ring of dual numbers*.

Definition 3.1.
(1) A *deformation functor* is a functor $F : \mathrm{Art}_k \to \mathrm{Set}$ such that $F(k)$ has one element.
(2) The *tangent space* of a deformation functor F is the set $TF = F(k[\varepsilon])$.

Examples 3.2.
(1) For a k-scheme X, let F be the functor

$$F(A) = \ker(\mathrm{Pic}(X \otimes A) \to \mathrm{Pic}(X)),$$

where $X \otimes A = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, A$ by abuse of notation. We proved in §2.4 that $TF = H^1(X, \mathscr{O}_X)$.
(2) Let X be a scheme over k, A an object of Art_k. By a *deformation* of X to A we mean a flat A-scheme $\widetilde{X}$ together with an isomorphism $\widetilde{X} \otimes k \simeq X$. In other words,

a deformation of X to A is a cartesian square

$$\begin{array}{ccc} \widetilde{X} & \longleftarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longleftarrow & \operatorname{Spec} k \end{array}$$

where the left vertical map is flat. Consider the functor

$$\operatorname{Def}_X : \operatorname{Art}_k \to \operatorname{Set}, \quad A \mapsto \{\text{deformations of } X \text{ to } A\}.$$

This is a deformation functor. We will see later that if X is *smooth* over k, there is a natural isomorphism $T\operatorname{Def}_X \simeq H^1(X, T_X)$ where T_X is the tangent sheaf of X.

(3) Let $\Lambda = k[[x_1, \ldots, x_n]]/I$ for some proper ideal I. Thus Λ is a complete local k-algebra with residue field k, and while it is not Artinian, it is a projective limit of objects of Art_k: $\Lambda = \lim \Lambda/\mathfrak{m}_\Lambda^n$. Consider the functor $F(A) = \operatorname{Hom}(\Lambda, A)$ (morphisms in the category of local k-algebras with residue field k). This is a deformation functor with tangent space $TF \simeq (\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2)^*$, the usual Zariski tangent space of Λ. A deformation functor of this form will be called *pro-representable*.

(4) Let Z be a k-scheme, $z \in Z(k)$ a k-point. Consider the functor $F : \operatorname{Art}_k \to \operatorname{Set}$ sending an algebra A to the set of morphisms $\operatorname{Spec} A \to Z$ such that the diagram

$$\begin{array}{ccc} \operatorname{Spec} k & \xrightarrow{z} & Z \\ \downarrow & \nearrow & \\ \operatorname{Spec} A & & \end{array}$$

commutes. This is a deformation functor, in fact isomorphic to the functor $\operatorname{Hom}(\hat{\mathscr{O}}_{Z,z}, -)$ (morphisms of local k-algebras endowed with a morphism to k). Thus if Z is noetherian and $k(z) = k$, the functor F is pro-representable. The tangent space TF is of course the tangent space to Z at z.

3.3. **Why should TF be a vector space?** The above examples suggest that often the set TF has a natural structure of a vector space over k. We observe first that the field k always acts on TF in a natural way: for $\lambda \in k$, let $m_\lambda : k[\varepsilon] \to k[\varepsilon]$ be the morphism sending ε to $\lambda \cdot \varepsilon$. This is an action of the multiplicative monoid of k on $k[\varepsilon]$, and $TF = F(k[\varepsilon])$ inherits this structure by functoriality. Trying to define a natural addition map $TF \times TF \to TF$, we are quickly led to the following question. Consider the morphism

$$\alpha : k[\varepsilon_1] \times_k k[\varepsilon_2] \longrightarrow k[\varepsilon], \quad \varepsilon_i \mapsto \varepsilon.$$

Note that fiber product over k is the categorical product in the category Art_k. Applying F, we obtain a diagram

$$\begin{array}{ccc} F(k[\varepsilon_1]\times_k k[\varepsilon_2]) & \xrightarrow{F(\alpha)} & F(k[\varepsilon]) \\ \downarrow & & \| \\ F(k[\varepsilon_1])\times F(k[\varepsilon_2]) & & TF \\ \| & \overset{?}{\nearrow} & \\ TF\times TF & & \end{array}$$

We see that if F commuted with the fiber product in question, we would get a map $a : TF \times TF \to TF$, a natural candidate for the addition map.

Lemma 3.4. *Suppose that F commutes with fiber products of the form $(-) \times_k k[\varepsilon]$. Then the maps $F(m_\lambda) : TF \to TF$ ($\lambda \in k$) together with the map $a : TF \times TF \to TF$ define a k-vector space structure on TF.*

Proof. Left as an exercise. □

4. Formal smoothness

Let $f : X \to Y$ be a locally finitely presented map of schemes. Recall that f is *smooth* if it is flat and $\Omega^1_{X/Y}$ is locally free of rank equal to the relative dimension of X over Y. If $Y = \operatorname{Spec} k$ for an algebraically closed field k, this just means that X is regular.

For us, the following criterion of smoothness (similar in spirit to the valuative criteria of separatedness and properness) will be more useful than the definition.

Proposition 4.1. *A locally finitely presented morphism $f : X \to Y$ is smooth if and only if, for every surjection $A' \to A$ whose kernel I is a nilpotent ideal, end every commutative square of solid arrows*

$$\begin{array}{ccc} \operatorname{Spec} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} A' & \longrightarrow & Y \end{array}$$

there exists a dotted arrow making the diagram commute.

Proof. We will only prove the $\Rightarrow$ implication (cf. [Har10, Proposition 4.4]). We can assume that $Y = \operatorname{Spec} S$, $X = \operatorname{Spec} R$ for a finitely presented S-algebra R. Choose a surjection $P \to R$ from a polynomial algebra P over S in finitely many variables, and let J be its kernel.

We can also assume that $I^2 = 0$ by considering inductively the sequence of surjections

$$A' = A'/I^n \to A'/I^{n+1} \to \ldots \to A'/I = A,$$

each with square-zero kernel.

Consider the diagram of solid arrows with exact rows below

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & P & \longrightarrow & R & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

Since P is a free algebra over S, there exists an S-algebra map g making the square on the right commute. Let $h : J \to I$ be its restriction to J. Then h induces $\overline{h} : J/J^2 \to I/I^2 = I$. On the other hand, since f is smooth, J/J^2 is the first term in short exact sequence

$$0 \to J/J^2 \to \Omega_{P/S} \otimes_S R \to \Omega_{R/S} \to 0$$

where the term on the right $\Omega_{R/S}$ is projective. Thus the sequence splits, and there exists an extension $u : \Omega_{P/S} \otimes_S R \to I$ of $\overline{h}$. Now the composition $\Omega_{P/S} \to I$ yields an S-derivation $v : P \to I$. Let $g' : P \to A'$ be the map $g'(x) = g(x) - v(x)$. Then one can check easily that g' is a homomorphism and $g'(J) = 0$. Thus g' descends to a homomorphism $R \to A'$, which gives a desired map $\operatorname{Spec} A' \to X$. □

This motivates the following definition.

Definition 4.2. Let $\tau : F \to G$ be a natural transformation between deformation functors. We say that τ is smooth if for every surjection $A' \to A$ in Art_k, and every commutative square of solid arrows

$$\begin{array}{ccc} \operatorname{Hom}(A, -) & \longrightarrow & F \\ \Big\downarrow & \nearrow & \Big\downarrow{\scriptstyle \tau} \\ \operatorname{Hom}(A', -) & \longrightarrow & G \end{array}$$

there exists a dotted arrow making the diagram commute, or in other words (by Yoneda), if the map

$$F(A') \longrightarrow F(A) \times_{G(A)} G(A')$$

is surjective. A deformation functor F is called smooth if the map to the trivial functor $G(A) = \{*\}$ is smooth, that is, if $F(A') \to F(A)$ is surjective for every surjection $A' \to A$.

Using a version of the above proposition, one can show that a morphism $\Lambda' \to \Lambda$ for Λ, Λ' as in REF gives a smooth morphism $\operatorname{Hom}(\Lambda, -) \to \operatorname{Hom}(\Lambda', -)$ if and only if $\Lambda \simeq \Lambda'[[y_1, \ldots, y_r]]$ for some $r \geq 0$, and that a morphism of varieties $f : X \to Y$ is smooth if and only if, for every closed point $x \in X$, the associated map $\operatorname{Hom}(\hat{\mathscr{O}}_{X,x}, -) \to \operatorname{Hom}(\hat{\mathscr{O}}_{Y,y}, -)$ is a smooth morphism of deformation functors.

References

[FG05] Barbara Fantechi and Lothar Göttsche, *Local properties and Hilbert schemes of points*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 139–178. MR 2223408

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[Har10] ———, *Deformation theory*, Graduate Texts in Mathematics, vol. 257, Springer, New York, 2010. MR 2583634